Altruism in General Equilibrium

Philip Trammell*

March 14, 2025

Abstract

Many people have not only preferences over their own consumption but also "altruistic" preferences over supply: e.g. enjoying meat but preferring that less be produced. Some have argued that altruism should not affect demand in competitive markets, but I generalize recent results that, though the price impact of a given purchase approaches zero as the economy is replicated, its supply impact stays significant, making altruism a rational first-order demand consideration even in a competitive setting. I then (1) characterize optimal demand adjustments for altruistic consumers, (2) define a general equilibrium concept that extends regular Walrasian equilibrium to the case in which some consumers are altruistic (their demand adjustments must be mutual best responses), and (3) show that a given altruistic preference can motivate an arbitrary demand adjustment in equilibrium. In particular, altruism can motivate consumers (a) to consume more of all goods which they would prefer in less supply and vice-versa—e.g. every consumer might buy more meat to reduce meat supply—and (b) not to exhaust their budgets, even if they prefer to increase both their own consumption and the supply of all goods.

1 Introduction

1.1 Motivation

Individuals' spending behavior affects not only the quantities of each good that they themselves consume, but also the total quantities of each good supplied. One may have preferences over all these quantities, and one may optimize one's purchasing behavior accordingly. This paper explores a model of general equilibrium in which individuals do so.

*Digital Economy Lab, Stanford University. 353 Jane Stanford Way, Stanford University, Stanford, CA 94305. Contact: trammell@stanford.edu. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Consider for example a consumer with concern for animal welfare. The consumer's utility is increasing in his own consumption of chicken and pork, holding supply fixed, but decreasing in the production of chickens and pigs. The latter preference—following Andreoni (1990) we will call it an altruistic preference—may motivate him to purchase less meat than he would otherwise. This is because, though a consumer's ability to impact equilibrium prices generically vanishes as the economy grows large (Roberts and Postlewaite, 1976), his ability to impact equilibrium supply does not. As the economy grows large and the price impacts of a given purchase shrink to zero, any price impact influences the decisions of a number of firms and other consumers that rises to infinity.

To see this, suppose a consumer changes his demand function in such a way that he buys one more unit of some good at any given price. Compare (a) the impact of this change on that good's equilibrium price and supply in a given economy with (b) the impact of this change on that good's equilibrium price and supply in a "doubled" economy with twice as many agents but an identical distribution of endowments, preferences, profit shares, and production technologies.

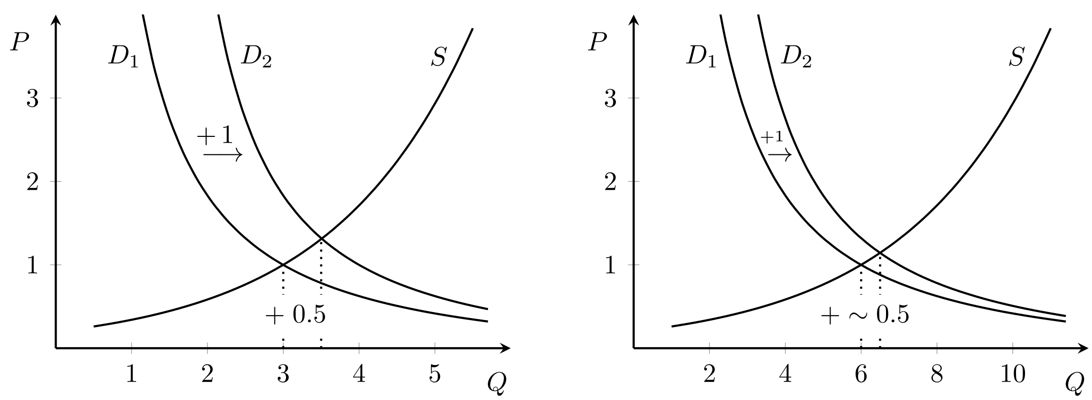

Fig. 1a: Impact on equilibrium quantity of a unit demand shock, original economy

Fig. 1b: Impact on equilibrium quantity of a unit demand shock, doubled economy

In the absence of the demand-shift, supply and demand in the doubled economy are precisely double what they are in the original economy. Since the price levels, original demand curve D_1, and supply curve S on the second graph are the same as on the first, the quantities on the horizontal axis are doubled. A one-unit increase in demand shifts the demand curve half as far in the second graph, so it has about half the original impact on the equilibrium price, but about the same impact on the equilibrium quantity.[1,2]

In light of this impact on supply, how should an animal-welfare-concerned consumer adjust his demands, relative to what they would be if he had no concern for animal welfare? If he prefers a dollar's reduction in chicken production to a dollar's reduction in pork production, he may naively think that his principal concern should be to reduce his own chicken consumption. If pork supply is more price-elastic and pork demand less price-elastic than that of chicken, however, this inclination may be misguided. Buying less chicken in this case causes the price to fall and the quantity demanded by other consumers to rise, with little net impact on the quantity of chicken consumed. Buying less pork, by contrast, generates a substantial decrease to the quantity of pork consumed. Cutting back on pork may thus be the higher priority.

Complicating matters further, our consumer must consider the impact of his purchases not only on the quantity of the good purchased, but on the quantities of all the goods he cares about. If buying less chicken causes other consumers to shift to chicken from other meat products, whereas buying less pork causes other consumers to shift to pork from vegetables, then cutting back on chicken may be the best policy after all.

In light of these complications, this paper characterizes optimal spending behavior by altruistic consumers, and equilibrium market behavior, in a competitive production economy where consumers may be altruistic. That is, we will study strategic consumer behavior given altruism in general equilibrium.

1.2 Related literature

Though this is the first paper to incorporate all three features (strategic demands, altruism, and general equilibrium effects), there are literatures on all three pairs of features in isolation.

First, there is an extensive literature exploring strategic behavior in general equilibrium without altruism. A central concern of this literature is to put the standard Walrasian model—in which agents are forbidden from strategic behavior and must act as price-takers—on a strategic footing. This is done by constructing a game in

[1] Kaufmann et al. (2024) offer survey evidence that many consumers believe that their consumption decisions affect supply (either one-for-one or, as in Figure 1, partially but significantly), and summarize the literature that many consumers do in fact have altruistic preferences over the aggregate supply of some goods strong enough to motivate non-negligible shifts in consumption.

[2] Note that this insight comes out only when we define competitive behavior in the usual way, as the limit of behavior across economies with finite but ever larger populations. Under the Aumann (1964) approach of modeling the population as a continuum, demand choices by individuals do not affect prices or supply.

which firms and/or individuals can choose their supply or demand correspondences, respectively, and equilibrium prices set the excess demands implied by these chosen correspondences equal to zero. Consider a sequence of such games set in ever larger "replicated economies", with each agent becoming an ever smaller part of the whole, and consider the sequence of Nash equilibria of these games. We can then ask under what circumstances, and in what sense, these Nash equilibria—these profiles of chosen supply and demand correspondences—can or must converge to the "Walrasian" profile in which agents all choose price-taking supply or demand correspondences.

Roberts and Postlewaite (1976) show that, in an exchange economy, the demand correspondences chosen in a sequence of Nash equilibria do not generally converge, even pointwise, to the Walrasian demand correspondences. They do find, however, that a consumer's impacts on equilibrium prices, equilibrium allocations, and her own utility by choosing her Walrasian demand correspondence fall to zero as the economy is replicated. It follows that if a consumer faces arbitrarily small costs—e.g. computational costs—to deviating from price-taking behavior, she will choose to act as a price-taker in a sufficiently large economy.

Otani and Sicilian (1990), studying a restriction to the same game, demonstrate that if consumers can only choose differentiable demand functions, there are sequences of Nash equilibria that, again, converge to Walrasian equilibria. Jackson and Manelli (1997) find that uncertainty about others' chosen strategies can also motivate consumers to adopt behavior that converges to fully price-taking behavior.

Though these papers all discuss exchange economies, they are perhaps the papers on strategic foundations for general equilibrium most relevant to this one, which will feature a production economy but strategic behavior only by consumers (not by firms). Nevertheless, there are many analogous well-known convergence results pertaining to production economies featuring strategic behavior by producers.[3] The message generally taken from this literature is that price-taking behavior reasonably approximates what one should expect in an economy in which prices result from the strategic behavior of consumers and/or firms with minimal market power. As we will see, and as suggested by the informal discussion of Section 1.1, the presence of altruistic consumers allows no such approximation.

A second literature explores general equilibrium with altruism in settings in which individuals are not permitted to act strategically.

Dufwenberg et al.'s (2011) "Other-Regarding Preferences in General Equilibrium" presents a model intended to capture equilibrium purchasing behavior in a production economy in which individuals may have altruistic preferences, in particular *other-regarding preferences*: preferences defined in part over the consumption baskets, utility levels, or budget sets achieved by other consumers. If Dufwenberg

[3] Most straightforwardly, the result that n firms engaged in Cournot competition approach perfectly competitive behavior as $n \to \infty$: see e.g. Varian (1992), p. 290.

et al. had allowed consumers to choose their demand functions strategically, their project would have encompassed the project of the present paper, since concern for the total production of various goods is a special case of concern for others' consumption baskets. Instead, however, Dufwenberg et al. maintain the classical assumption that each individual is *fully* price-taking, in the sense that she acts as if her own purchases have no impact on prices and thus no impact on aggregate production, others' budget sets, and so on. They thus conclude that an individual's other-regarding preferences have no impact on her purchasing behavior unless they are not separable from her self-regarding preferences: for example, if she prefers brown bread to white bread if and only if her neighbor has two cars.

A similar "fully price-taking" approach to other-regarding preferences, or preferences over supply levels, is taken by Kreps (1990) (p. 203), Ellickson (1994) (Section 7.3), Sobel (2009), and others.

The approach taken in this literature is motivated by the observation from the literature on strategic foundations for general equilibrium that, in large economies and in the absence of altruism, relatively mild assumptions can guarantee that strategic behavior differs little (or produces outcomes that differ little) from price-taking behavior. In contrast, the present paper is centered on the insight that, as noted above, altruism will typically motivate strategic consumer behavior that differs substantially from price-taking behavior, however large the economy relative to each consumer.

Finally, three strands of literature explore the implications of strategic demand behavior by altruists, but do so in a partial equilibrium setting.

The first and most important is a standalone paper by Kaufmann et al. (2024), who develop a model essentially identical to that presented here but in a highly restricted environment.[4] Kaufmann et al. study the special case of an economy with only two goods, a linear supply curve, and symmetric consumers who have quasilinear utility in consumption and are indifferent about the supply of the quasilinear good.[5] In such a setting the quasilinear good is effectively money, so the model amounts to a partial equilibrium analog to the general equilibrium model explored here.

These restrictions allow the authors to illustrate some of the conceptual subtleties of strategic consumer behavior more straightforwardly, and to detail the economic and policy implications of altruism by consumers in the restricted setting. Their results however offer little guidance to a consumer hoping to achieve a given impact on production quantities in a world of interconnected markets. As outlined below, a contribution of the present paper is to demonstrate that many economic implications of altruism which can be identified in the partial equilibrium case do not generalize.

The second strand is the literature on the private provision of public goods.

[4]That paper and this one were written roughly concurrently.

[5]In one setting they consider three goods, of which two are in a certain sense perfect substitutes.

Bergstrom et al. (1986) study equilibrium spending behavior by public good providers in light of crowd-out issues like those discussed above. The model we will consider is thus conceptually related to the model introduced by Bergstrom et al. and developed throughout the subsequent literature on the private provision of public goods and bads.

Our model, however, will vary from the models typically explored by that literature in two interrelated ways. First, whereas models of public good provision games typically treat prices as exogenous, we will allow goods' prices to be determined endogenously by the starting endowments, production technologies, and quantities of other goods purchased. Second, we will allow some individuals to have preferences that depend only on their own consumption baskets and not at all on total supply levels. (Without the first variation—i.e. the introduction of endogenous prices—fully "selfish" individuals have no impact on the game played among those who do care about total supply levels. Selfish individuals can therefore safely be excluded from the model.)

The final strand is the literature, apparently confined so far entirely to agricultural economics, on "equilibrium displacement models", or EDMs. (See Wohlgenant (2011) for a review.) EDMs generate predictions about the impact that purchases of a particular good have on that good's equilibrium supply level. Norwood and Lusk (2011), for instance, estimate the price elasticities of supply and demand for various animal products and, from these estimations, calculate the extent to which marginal purchases of a given animal product change the equilibrium quantity supplied of that product. Wilkinson (2022) uses a similar EDM analysis in arguing that consumers are morally obligated to account for the market externalities of their purchases: i.e. the impacts that their purchases have on prices, and the impacts of these price-shifts on others' purchases and ultimately others' welfare.

The two pieces cited above—along with most of the EDM literature—only consider the impact of purchasing a certain good on the equilibrium supply of that good. In effect, they consider only the equilibrium supply-shift presented graphically in Section 1.1. For our purposes, such an analysis is relevant only in the event that cross-price elasticities of demand and supply are zero across goods over whose supply a consumer has altruistic preferences, and complete only under still stronger conditions. Goods about which one has altruistic preferences, however, are often highly substitutable (for consumers and other producers) with goods about which one has similar altruistic preferences. Recall the example of Section 1.1: a consumer who reduces her consumption of chicken on altruistic grounds is likely to be concerned with decreasing the production not only of chicken but also of other meat products. Partial equilibrium analyses like the two cited above may thus be highly misleading.

An obscure but especially relevant step toward generality, within the EDM literature, is taken by Gardner (1987). Gardner's framework allows for an analysis of multiple goods simultaneously, in which changes to an individual's demand for one good affect the equilibrium prices and supply levels of other goods under consideration. His framework nonetheless falls short of a general equilibrium framework in two ways. First, it can serve only as a partial guide to the altruist, as it breaks down when the consumer attempts to account for her impacts on all markets at once. Second, it serves *only* as a guide to the individual altruist. It does not address how to define or characterize a notion of economic equilibrium in a world with multiple (let alone all) consumers choosing demands in light of altruistic preferences. It is thus in no way continuous with the literature on public good games, referenced above.

The general equilibrium analysis presented here closes this gap.

1.3 Outline

Section 2 introduces the model: a game in which consumers strategically choose demand functions to maximize preferences over (i) own consumption and (ii) supply. Section 3 characterizes the equilibria of the game, proposes a natural equilibrium refinement, and identifies conditions under which an equilibrium is guaranteed to exist.

Section 4 explores the often counterintuitive relationship between altruistic preferences and optimal demand adjustments in the general equilibrium setting of Sections 2 and 3. The section thus highlights the importance to the altruist of accounting for cross-market effects, and difficulty for small consumers—and, by extension, public good contributors—of predictably achieving desired effects on quantities produced.

Section 5 concludes with a discussion of the value of better tools for identifying such effects and a summary of how the analysis above might be extended.

2 Model

The model is one in which consumers *choose demand functions* in order to satisfy their preferences over both aggregate supply and their own consumption. We introduce the model in two stages. First, we define the economic environment within which the game, in which individuals choose their demand functions, will be played. This will allow us to define the economic outcomes that obtain given a profile of demand functions. Next we introduce preferences and define the game itself, along with the equilibrium concept.

2.1 Economic environment

Production and prices

There are $L \geq 2$ goods, indexed by ℓ or k.

There are I individuals, indexed by i, with endowment vectors $e^i > 0$.[6] The vector of total endowments is e.

A price vector p is of dimension L, with $p > 0$. Given prices p, the vector of net production $y(p)$ maximizes $p \cdot y$ within a production possibility set Y:

$$y(p) \equiv \underset{y \in Y}{\text{argmax}}\ p \cdot y. \tag{1}$$

Y is compact, and its efficient frontier is strictly concave, ensuring that $y(p)$ exists and is unique. Y satisfies $y \geq -e\ \forall y \in Y$, ensuring that supply $s(p) \equiv e + y(p)$ is non-negative. It follows from (1) that $y(\cdot)$ and thus $s(\cdot)$ are homogeneous of degree (h.o.d.) 0.

Definition 1. *An environment E is a tuple of endowments $\{e^i\}_{i=1}^I$, aggregate profit shares $\{\theta^i\}_{i=1}^I$, and a production function $y(\cdot)$ such that*

$$e^i > 0\ \ \forall i; \tag{2}$$

$$\theta^i \geq 0\ \ \forall i,\ \ \textstyle\sum_{i=1}^I \theta^i = 1;\ \textit{and} \tag{3}$$

$$y(p), \text{ defined on } \mathbb{R}^L_{++}, \text{ maximizes } p \cdot y \text{ subject to } y \in Y, \text{ where} \tag{4}$$
$$Y \text{ is compact, its efficient frontier is strictly concave, and } y \geq -e\ \forall y \in Y.$$

(An economy is defined below as an environment and a profile of demand functions.)

We have assumed that production y maximizes $p \cdot y$, not that production maximizes *profits*. We will define profits below, finding that if consumers exhaust their budgets, aggregate profits equal $p \cdot y$. We will not in general assume that all consumers exhaust their budgets, however, and if some consumers do not exhaust their budgets, aggregate profits either exceed $p \cdot y$ or are undefined. Nevertheless, we will see that the production vector y that maximizes $p \cdot y$ also maximizes aggregate profits (when these are defined).

Demand

Aggregate profits (hereafter, profits) are denoted π. Profits are divided among consumers according to shares $\theta^i \geq 0$. Given prices p and profits π, i's budget equals $p \cdot e^i + \theta^i \pi$.

Individual i chooses a demand function, defined over positive prices and non-negative profits: $x^i(p, \pi)$. (Demand functions will be chosen strategically to satisfy preferences, as discussed in the next section. For now we will take them as given.) The aggregate demand function, or sum of individual demand functions, is denoted $x(\cdot)$.

[6] When clear, we will let 0 denote a column vector of zeroes. When it is necessary to distinguish this vector from the scalar, we will denote the L-dimensional column vector of zeroes by $\mathbb{0}_L$. Given equal-length vectors b, c, we will use the notation that $b \geq c$ if $b_\ell \geq c_\ell\ \forall \ell$; $b > c$ if $b \geq c$ and $b \neq c$; $b \gg c$ if $b_\ell > c_\ell\ \forall \ell$; and $b \propto c$ if $b = \kappa c$ for some real κ.

Definition 2. *A function $x^i(p,\pi)$ from $\mathbb{R}^L_{++} \times \mathbb{R}_+$ to $\mathbb{R}^L_+$ is an* <u>*admissible demand function*</u> *for i in environment E if $x^i(\cdot)$ is h.o.d. 0, feasible in the sense that*

$$p \cdot x^i(p,\pi) \leq p \cdot e^i + \theta^i \pi \ \ \forall (p,\pi) \in \mathbb{R}^L_{++} \times \mathbb{R}_+, \tag{5}$$

and non-decreasing in π.

We will denote the set of admissible demand functions for i in E by $\mathbb{A}^i(E)$ and the set of admissible demand function profiles in E by $\{\mathbb{A}^i(E)\}$. An admissible aggregate demand function is the sum of an admissible demand function profile, and the set of admissible aggregate demand functions is denoted $\mathbb{A}(E)$.

We deliberately impose only minimal restrictions on demand functions here in order to emphasize the point that individuals may choose their demand functions strategically, and we have not yet determined what demand functions (if any) are incompatible with optimization in the current context. Further restrictions on demand functions will be introduced below where appropriate.

Profits

The aggregate budget constraint is $p \cdot e + \pi$. If profits π are defined to equal $p \cdot y$, as is conventional, then the aggregate budget constraint is $p \cdot (e + y) = p \cdot s$. So if some people choose not to exhaust their budgets, the value of demand $p \cdot x$ must be *less* than the value of supply $p \cdot s$, and markets cannot clear.

Here, we do not assume local nonsatiation. We allow for the possibility that, in some circumstances and for some altruistic preferences individuals might have, an individual maximizes his utility by choosing a demand function in which he does not always exhaust his budget. We therefore use a generalized definition of profits commonly used in the event that individuals may be satiated.[7]

Given prices $p \gg 0$ and an admissible aggregate demand function $x(\cdot)$, profits equal

$$\pi(p, x(\cdot)) \equiv \underline{\pi \text{ such that } p \cdot x(p,\pi) = p \cdot (e + y(p))} \tag{6}$$

if the expression above is defined (and are undefined otherwise). Observe that by the feasibility of the individual demand functions,

$$p \cdot x(p, p \cdot y(p)) \leq p \cdot (e + y(p)). \tag{7}$$

If defined, therefore, π weakly exceeds $p \cdot y(p)$.

For intuition, suppose that there is a single good. Each individual has an endowment of 1 and sells it to a firm which they all have equal shares in, and the firm

[7]Konovalov (2005) offers a review of the literature using this approach, at least as of 2005. The resulting economic equilibria are called "dividend equilibria" or "Walrasian equilibria with slack". We will use the latter term, to emphasize that these equilibria generalize the familiar notion of Walrasian equilibrium.

sells it back at the same price. If all exhaust their budgets, then markets clear and profits equal zero. But if one person buys nothing, then instead of simply saying that markets do not clear, we can observe there is some profit level that would clear them. If the firm's profit equals $I/(I-1)$, then each individual receives a dividend of $1/(I-1)$. The $I-1$ spenders spend their dividends, and the 1 non-spender does not, so the $I-1$ spenders collectively buy the 1 extra unit that the non-spender sold and did not buy back.

The market-clearing profit level of $I/(I-1)$ can be conceived of iteratively. One person buys nothing, so at the first step the firm makes a profit of 1: there is one unit of "leftover inventory". So each individual receives a dividend of $1/I$. But since only $I-1$ of them spends this dividend, this payout only clears out $(1-I)/I$ units of inventory, leaving $1/I$ left. This gets paid out in equal proportions again, leaving $1/I^2$ left over, and so on. Summing this series yields $1+1/I+1/I^2+... = I/(I-1)$. Defining profits always to equal $p \cdot y$ (here, $=0$) would imply that, if anyone does not exhaust his budget, firms remain in possession of unsold goods of value of $p \cdot (s-x)$ (here, $=p$), leaving them to rot on the shelves, though some of their owners would have liked to consume them.

An admissible demand function profile does not guarantee that profits are well-defined: $\pi(p, x(\cdot))$ may not exist or not be unique. Profits are however always defined if aggregate demand is nonsatiated, meaning that $p \cdot x(p, \pi)$ strictly increases without bound in π. Since $p \cdot x^i(p, \pi)$ weakly increases in π for all i, nonsatiation is guaranteed as long as $p \cdot x^i(p, \pi)$ strictly increases without bound in π for some i.

When individuals do exhaust their budgets,

$$p \cdot x(p, p \cdot y) = p \cdot e + p \cdot y = p \cdot s. \tag{8}$$

Profits then are always defined and equal $p \cdot y$, as usual. Also, even in the more general setting of (6), given that profits are uniquely defined, because $p \cdot x(p, \pi)$ weakly increases in π, $y(p)$ maximizes profits iff $y(p)$ maximizes $p \cdot s$ and thus $p \cdot y$.

Note that instead of introducing a set of profit-maximizing firms, we are positing that production maximizes profits subject to a single, abstract production technology. If we modeled firms explicitly, we would have to model how their behavior responded to the diverse preferences of their owners, not all of whom in this setting will care only about profit maximization. We would likewise have to explore the possibility that some distributions of firm ownership are incompatible with equilibrium. Individuals with preferences regarding the production levels of some goods might, for instance, want to concentrate their capital holdings in a few relevant firms so as to wield influence as "activist investors". To restrict our focus to the equilibrium consequences of market purchases, we will therefore simply assume that production proceeds along profit-maximizing lines.

Economic equilibria

Definition 3. *Given an admissible and nonsatiated aggregate demand function* $x(p, \pi)$ *and a supply function* $s(p)$, *implicit demand equals*

$$\chi(p) \equiv x\big(p, \pi(p, x(\cdot))\big), \tag{9}$$

and excess demand equals

$$z(p) \equiv \chi(p) - s(p). \tag{10}$$

Let

$$\mathcal{I} \equiv [I_{L-1}, \mathbb{0}_{L-1}] : \tag{11}$$

the $(L-1) \times L$ matrix whose first $L-1$ columns constitute the $(L-1) \times (L-1)$ identity matrix and whose last column is the zero vector.

Definition 4. *Given price vector* $p \gg 0$, *the corresponding normalized price vector equals*

$$\hat{p} \equiv \mathcal{I}p/p_L. \tag{12}$$

The L^{th} entry of p/p_L is always 1, and left-multiplication by $\mathcal{I}$ removes this redundant entry. In general, we will use the "hat" symbol to denote normalization from L to $L-1$ dimensions in this way, and we will use the upside-down hat (or "check") to denote expansion from $L-1$ to L dimensions, so that given normalized price vector $\hat{p}$,

$$\check{\hat{p}} \equiv (\hat{p}, 1) \ \big(= p/p_L, \text{ if } p \text{ is given}\big); \tag{13}$$

and normalized supply, excess demand, etc. functions are

$$\begin{aligned} \hat{s}(\hat{p}) &\equiv \mathcal{I}s(\check{\hat{p}}), \\ \hat{z}(\hat{p}) &\equiv \mathcal{I}z(\check{\hat{p}}), \text{ etc.} \end{aligned} \tag{14}$$

Definition 5. *Price vector* $\bar{p} \gg 0$ *is a Walrasian equilibrium with slack (WES) of environment* E *and aggregate demand function* $x(\cdot) \in \mathbb{A}(E)$ *[or demand function profile* $\{x^i(\cdot)\} \in \{\mathbb{A}^i(E)\}$*] if* $z(\bar{p}) = 0$.
Normalized price vector $\hat{\bar{p}}$ *is a normalized WES (NWES) of* $(E, x(\cdot))$ *if* $\hat{z}(\hat{\bar{p}}) = 0$.

Since in general profits may be undefined, so may implicit demand, and so may excess demand. If $\bar{p}$ is a WES of $(E, x(\cdot))$, therefore, it follows that $\pi(\bar{p}, x(\cdot))$ is defined.

Given an environment E, an aggregate demand function $x(\cdot) \in \mathbb{A}(E)$, and a WES $\bar{p}$ of $(E, x(\cdot))$, we will write as shorthand $\bar{\pi} \equiv \pi(\bar{p}, x(\cdot))$.

Definition 6. *A WES $\bar{p}$ of environment E and aggregate demand function $x(\cdot) \in \mathbb{A}(E)$ is a* <u>*regular WES (RWES)*</u> *if (a) the Jacobian $D\hat{z}(\hat{\bar{p}})$ is nonsingular and (b) $s(\cdot)$ is locally $\mathcal{C}^1$ around $\bar{p}$.*

We will refer to a normalized regular WES as an NRWES. The regularity of a Walrasian equilibrium is typically defined to be condition (a) in isolation (see e.g. Mas-Colell et al. (1995), p. 591). We define regularity more strongly here because we will only ever require (a) in conjunction with (b).

Definition 7. *Given a WES $\bar{p}$ of environment E and demand function profile $\{x^i(\cdot)\}$, the demand function profile is* <u>*locally regular*</u> *if each $x^i(\cdot)$ is locally $\mathcal{C}^1$ and $\frac{\partial}{\partial \pi}\big(p \cdot x(p, \pi)\big) > 0$ around $(\bar{p}, \bar{\pi})$.*

2.2 Strategic demand function choice

It may be helpful to open with a summary of the strategic interaction we will define precisely in this section.

An environment E obtains, and each individual i has a utility function $u^i(x^i, s)$, defined over his own consumption and aggregate supply. Each i chooses a demand function $x^i(\cdot) \in \mathbb{A}^i(E)$. In conjunction with the environment's supply function $s(\cdot)$, the demand function profile $\{x^i(\cdot)\} \in \{\mathbb{A}^i(E)\}$ is compatible with a (perhaps empty) set of RWES $\bar{p}$, which given $\{x^i(\cdot)\}$ uniquely determine a profit level, aggregate supply, and individual demands. A demand function profile $\{x^i(\cdot)\}$ and a compatible RWES $\bar{p}$ will be called an equilibrium if, for each i and each demand function $\tilde{x}^i(\cdot)$ admissible for i in E (subject to a smoothness condition),

- there is a unique normalized price vector $\hat{p}$ near $\hat{\bar{p}}$ that is an NRWES compatible with $\{x^{-i}(\cdot), \tilde{x}^i(\cdot)\}$ and $s(\cdot)$; and
- the u^i that obtains given $(\hat{p}, \{\tilde{x}^i(\cdot), x^{-i}(\cdot)\})$ is, in the limit as the economy is replicated and i's market power vanishes, no greater than the u^i that obtains given $(\hat{\bar{p}}, \{x^i(\cdot)\})$.

Preferences

Individual i has a utility function $u^i(\cdot)$ representing her preferences over $(x^i, s) \in \mathbb{R}^{2L}_+$. We will assume that her preferences are additively separable across x^i and s, so that it is meaningful to refer independently to <u>consumption preferences</u> ranking consumption baskets x^i and <u>altruistic preferences</u> ranking supply baskets s, and that she has a utility function of the form

$$u^i(x^i, s) = v^i(x^i) + w^i(s). \tag{15}$$

Her <u>consumption utility function</u> $v^i(\cdot)$ is continuous. Her <u>altruistic utility function</u> $w^i(\cdot)$ is $\mathcal{C}^1$.

We do not here impose the standard assumptions that $v^i(\cdot)$ is locally nonsatiated (or even non-decreasing). In particular, we do not rule out the possibility that $v^i(\cdot)$ is independent of x^i_ℓ for some or all ℓ. Thus, though the application to ethical consumerism guides our running interpretation of the model, the framework allows for individuals to purchase some goods entirely because of the altruistic impacts of such purchases—i.e. as acts of philanthropy. It likewise allows for the interpretation that some agents are consequentialist philanthropic foundations, exhibiting preferences defined exclusively over supply.

A utility function profile $\{u^i(\cdot)\}_{i=1}^I$ is admissible if $u^i(\cdot)$ satisfies the conditions above for all i.

Definition 8. *An economy $\mathcal{E}$ is an admissible utility function profile $\{u^i(\cdot)\}$ and an environment E.*

Given an economy $\mathcal{E} = (\{u^i(\cdot)\}, E)$, an (individual/aggregate/etc.) demand function is "admissible in $\mathcal{E}$", denoted e.g. $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E})$, if the object in question is admissible in E. Likewise, $\bar{p}$ is a [N][R]WES of $(\mathcal{E}, x(\cdot))$ iff it is a [N][R]WES of $(E, x(\cdot))$.

WES displacement

Definition 9. *Given economy*

$$\mathcal{E} = \big(\{u^i(x^i, s)\}_{i=1}^I, \{e^i\}_{i=1}^I, \{\theta^i\}_{i=1}^I, y(p)\big),$$

i's n-replicated utility function equals

$$u^{i(n)}(x^i, s) \equiv v^i(x^i) + nw^i(s/n). \tag{16}$$

The n-replicated economy equals

$$\mathcal{E}^{(n)} \equiv \big(\{u^{[i \bmod I](n)}(x^i, s)\}_{i=1}^{nI}, \{e^{i \bmod I}\}_{i=1}^{nI}, \{\theta^{i \bmod I}/n\}_{i=1}^{nI}, ny(p)\big). \tag{17}$$

n-replicated (individual/aggregate/etc.) demand functions equal

$$\begin{aligned} x^{i(n)}(p, \pi) &\equiv x^i(p, \pi/n), \\ x^{(n)}(p, \pi) &\equiv nx(p, \pi/n), \\ x^{-i(n)}(p, \pi) &\equiv x^{(n)}(p, \pi) - x^{i(n)}(p, \pi), \text{ and} \\ \{x^i(p, \pi)\}^{(n)} &\equiv \{x^{i \bmod I}(p, \pi/n)\}_{i=1}^{nI}. \end{aligned} \tag{18}$$

n-replicated profits equal

$$\begin{aligned} \pi^{(n)}(p, x(\cdot)) &\equiv \pi : p \cdot x(p, \pi) = np \cdot s(p) \\ &= n\big(\pi : p \cdot x(p, \pi)/n = p \cdot s(p)\big) \\ &= n\pi(p, x(\cdot)/n). \end{aligned} \tag{19}$$

In (16), the utility function of i and her "replicas" in the n-replicated economy is defined so that replication diminishes i's market power without changing either the location of supply in her altruistic utility function or her preferences over marginal changes in supply.

If we adopt

$$u^{i(n)}(x^i, s) = u^i(x^i, s),$$

then replication does not change i's economic circumstances only by diminishing her market power, but also by changing the value of a variable over which her preferences are defined, namely supply. For instance if $w^i(s)$ is strictly decreasing and strictly concave in s_ℓ—as it might be if the production of good ℓ is emissions-intensive and i considers environmental damages to be convex in emissions—then the larger quantity of emissions in the replicated economy increases the intensity of i's preference for marginal emissions reductions.

If we adopt

$$u^{i(n)}(x^i, s) = u^i(x^i, s/n),$$

so that i in the n-replicated economy simply acts as if supply were $1/n$ times as large as it is, then the problem above is resolved, but a new one is introduced. An altruistic consumer's concern in a given economy, large or small, is with the impact of a potential purchase on s itself—not its impact on s/n, an impact which is of course only $1/n$ times as large. The impact of a given purchase on s is asymptotically constant and generically nonzero as an economy is replicated (as shown formally in the proof of Proposition 3 below, and illustrated by Figure 1), whereas the impact of a given purchase on s/n falls to zero.

Utility function (16) resolves both issues. Note that if each $w^i(\cdot)$ is CRS, our replication is defined in the usual way, without any modification to individuals' utility functions.

Proposition 1. ***Preservation of WES under replication***
Given economy $\mathcal{E}$ and aggregate demand function $x(\cdot) \in \mathbb{A}(\mathcal{E})$, $\bar{p}$ is a [N][R]WES of $(\mathcal{E}, x(\cdot))$ iff $\bar{p}$ is a [N][R]WES of $(\mathcal{E}^{(n)}, x^{(n)}(\cdot))$ $\forall n \geq 1$.

Proof. See Appendix B.1. □

Proposition 2. ***Local RWES under demand shifts***
Let $\bar{p}$ be an RWES of economy $\mathcal{E}$ and locally regular demand function profile $\{\bar{x}^i(\cdot)\} \in \{\mathbb{A}^i(\mathcal{E})\}$. Then there exists an $\epsilon > 0$ and an $\underline{n} \geq 1$ such that

a. for any i and $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E})$ that is $\mathcal{C}^1$ around $(\bar{p}, \bar{\pi})$, for $n \geq \underline{n}$ there exists an NRWES $\hat{p}$ of $(\mathcal{E}^{(n)}, \bar{x}^{-i(n)}(\cdot) + x^{i(n)}(\cdot))$ within the ϵ-neighborhood of $\hat{\bar{p}}$; and

b. within the ϵ-neighborhood of $\hat{\bar{p}}$, $\hat{p}$ is the unique NWES.

Proof. See Appendix B.2. □

Definition 10. *An economy $\mathcal{E}$ is <u>large</u> with respect to $\{\bar{x}^i(\cdot)\} \in \{\mathbb{A}^i(\mathcal{E})\}$ and WES $\bar{p}$ of $(\mathcal{E}, \bar{x}(\cdot))$ if Proposition 2 holds for $(\mathcal{E}, \bar{x}(\cdot), \bar{p})$ given $\underline{n} = 1$.*

Equilibrium concept

Given an economy $\mathcal{E}$, demand functions $\{\bar{x}^i(\cdot)\} \in \{\mathbb{A}^i(\mathcal{E})\}$, and an RWES $\bar{p}$ such that each $\bar{x}^i(\cdot)$ is locally $\mathcal{C}^1$ around $(\bar{p}, \bar{\pi})$, choose n such that $\mathcal{E}^{(n)}$ is large with respect to $\{x^i(\cdot)\}$. Denote the locally unique NWES of $(\mathcal{E}^{(n)}, \bar{x}^{-i(n)}(\cdot) + x^{i(n)}(\cdot))$, for some locally $\mathcal{C}^1$ $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E})$, by

$$\hat{p}_{\bar{p}}^{(n)}\big(x^i(\cdot)\big).$$

Then, given locally $\mathcal{C}^1$ demand function $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E})$, let $u_{\bar{p}}^{i(n)}\big(x^i(\cdot)\big)$ denote the utility i achieves by demand function $x^{i(n)}(\cdot)$ in the replicated economy and demand function profile:

$$u_{\bar{p}}^{i(n)}\big(x^i(\cdot)\big) \equiv u^i\left(x^{i(n)}\Big(\check{\hat{p}}_{\bar{p}}^{(n)}\big(x^i(\cdot)\big), \pi^{(n)}\big(\check{\hat{p}}_{\bar{p}}^{(n)}\big(x^i(\cdot)\big), \bar{x}^{-i(n)}(\cdot) + x^{i(n)}(\cdot)\big)\Big), s\Big(\check{\hat{p}}_{\bar{p}}^{(n)}\big(x^i(\cdot)\big)\Big)\right). \quad (20)$$

The expression appears complex because it captures the several channels through which a choice of demand function $x^i(\cdot)$ in economy n affects i's utility, but each channel is straightforward. In economy n, i's utility, given demand function $x^i(\cdot)$, depends in the usual way on his consumption x^i and supply s. The former depends on prices and profits, the latter only prices. Profits depend on prices and $x^i(\cdot)$. Prices depend on $x^i(\cdot)$.

We can now define a competitive equilibrium concept suited to the case in which some individuals are altruistic.

Definition 11. *Let $\bar{p}$ be an RWES of economy $\mathcal{E}$ and locally regular demand function profile $\{\bar{x}^i(\cdot)\} \in \{\mathbb{A}(\mathcal{E})\}$. Then $(\bar{p}, \{\bar{x}^i(\cdot)\})$ is a competitive equilibrium with supply externalities (CESE) of $\mathcal{E}$ if, for all i and all locally $\mathcal{C}^1$ $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E})$,*

$$\lim_{n\to\infty}\left[u_{\bar{p}}^{i(n)}(x^i(\cdot)) - u_{\bar{p}}^{i(n)}(\bar{x}^i(\cdot))\right] \leq 0. \quad (21)$$

So $(\bar{p}, \{\bar{x}^i(\cdot)\})$ is a CESE if, in response to $\bar{x}^{-i}(\cdot)$, each i chooses an admissible and locally $\mathcal{C}^1$ $x^i(\cdot)$ that would be optimal for i if the economy were "infinitely large" relative to i. In effect, $x^i(\cdot)$ must be a demand function i is content to choose if he acts as a price-taker (because his impacts on prices are infinitesimal) but not as a quantity-taker (because his impacts on prices affect the behavior of infinitely many other agents, resulting in positive quantity effects even in the limit).

After addressing certain technicalities, the strategic interaction described in this section can be expressed as a normal form game, and CESE may be understood as a weakening of Nash equilibrium. This is discussed in Appendix A.1.

2.3 Summary

The components of the model are summarized below. The number of goods $L \geq 2$ and individuals I is fixed throughout. Demand and utility functions are assumed to satisfy the "admissibility" conditions defined below unless otherwise stated.

Environment	(1a) Profiles of endowments $\{e^i\} > 0$ and (1b) profit shares $\theta^i \geq 0$ summing to 1; and (2) a production function $y(p)$ maximizing $p \cdot y$ within a production set Y (which satisfies certain conditions).
Quantities until the first horizontal line are defined given an environment.	
[Admissible] demand function	A function $x^i(p, \pi)$ that is h.o.d. 0, feasible ($p \cdot x^i \leq p \cdot e^i + \theta^i \pi$), and non-decreasing in π.
Supply	Endowments plus production: $s(p) \equiv e + y(p)$.
Profits	Given prices p and agg. dmd. fcn. $x(p, \pi)$, π sets $p \cdot x(p, \pi) = p \cdot s(p)$. (Defined if $x(p, \pi)$ increases without bound in π.)
Implicit demand	Given $x(p, \pi)$, $\chi(p)$ expresses agg. dmd. as a function of prices, incorporating how prices determine profits.
WES	Prices p setting excess demand $\chi(p) - s(p)$ equal to 0.
RWES	A WES at which $s(\cdot)$ is $\mathcal{C}^1$ and the Jacobian of "normalized" excess dmd. (for $L - 1$ goods, in their prices) is nonsingular.
Locally regular dmd. fcn. profile	Given WES p, $\{x^i(\cdot)\}$ such that each $x^i(\cdot)$ is locally $\mathcal{C}^1$ and $\frac{\partial}{\partial \pi}(p \cdot x(p, \pi)) > 0$ at $(p, \pi(p, x(\cdot)))$.
[Admissible] utility function	$u^i(x^i, s) = v^i(x^i) + w^i(s)$, with $v^i(\cdot)$ continuous and $w^i(\cdot)$ $\mathcal{C}^1$.
Economy	(1) An environment and (2) a utility function profile.
CESE	Given an economy, (1) an RWES and (2) a locally regular dmd. fcn. profile s.t. any utility gain to any i of deviating to another locally $\mathcal{C}^1$ dmd. fcn. vanishes as the economy is replicated.

3 Equilibrium

3.1 Characterization

Given an economy $\mathcal{E}$, define the following terms:

$$\begin{aligned}
\delta(p, x(\cdot)) &\equiv \text{The gradient of the aggregate Engel curve at } \big(p, \pi(p, x(\cdot))\big), \text{ i.e. } \nabla_\pi x\big(p, \pi(p, x(\cdot))\big). \\
G(p, x(\cdot)) &\equiv \text{The generalized inverse } G \text{ of } -Dz(p), \text{ where } z(\cdot) \text{ is implied by } (\mathcal{E}, x(\cdot)), \text{ with } G\delta(p, x(\cdot)) = \mathbb{0}_L \text{ and whose bottom row equals } \mathbb{0}_L^T. \qquad (22) \\
\psi^i(p, x(\cdot)) &\equiv \big(Ds(p)\, G(p, x(\cdot))\big)^T \nabla w^i(s(p)). \qquad (23)
\end{aligned}$$

We can now characterize CESE relatively simply.[8]

Proposition 3. ***Characterization of CESE***
Let $\bar{p}$ be an RWES of economy $\mathcal{E}$ and locally regular demand function profile $\{\bar{x}^i(\cdot)\} \in \{\mathbb{A}(\mathcal{E})\}$. Then $(\bar{p}, \{\bar{x}^i(\cdot)\})$ is a CESE of $\mathcal{E}$ iff

$$\bar{x}^i(\bar{p}, \bar{\pi}) \in \operatorname*{argmax}_{x^i} \Big(v^i(x^i) + \psi^i(\bar{p}, \bar{x}(\cdot)) \cdot x^i\Big) \;\Big|\; \bar{p} \cdot x^i \leq \bar{p} \cdot e^i + \theta^i \bar{\pi} \;\; \forall i. \qquad (24)$$

Proof. See Appendix B.3. □

An intuition for the result is as follows.

Starting from a WES $\bar{p}$, suppose i adjusts his demand around $\bar{p}$ by a just-feasible dx^i: i.e., by a value dx^i satisfying $\bar{p} \cdot dx^i = 0$, or $dx^i \perp \bar{p}$. The impact of this demand-adjustment on prices will, given the appropriate smoothness conditions and given that i is a negligible part of the economy, be linear in dx^i, and can therefore be represented by $dp = G dx^i$ for some matrix G. G must satisfy

$$-Dz\, G\, dx^i = dx^i \;\; \forall dx^i \perp \bar{p}. \qquad (25)$$

That is, the equilibrium price impact dp of demand-shift dx^i must motivate shifts in supply ($Ds\, dp$) and in others' demands ($D\chi\, dp$) such that the gap between total supply and others' demands changes by precisely dx^i. Recall that $-Dz = Ds - D\chi$.

G must therefore be a generalized inverse of $-Dz$. Furthermore, the equilibrium price impact of a demand-shift along the gradient of the aggregate Engel curve must be zero: such a shift can be precisely accommodated by changing the profit rate, and thus others' demands, without affecting prices or supply. We thus have $G\delta = \mathbb{0}_L$. Finally, we make G uniquely defined by restricting the space of price vectors under

[8] Since economic price substitution data is often reported in terms of elasticities, an analog to (23) in elasticity terms is offered in Appendix A.2.

consideration to those whose L^{th} entries equal 1 (or, more precisely, we restrict the space of marginal price-*changes* dp to those with $dp_L = 0$, so that p_L remains fixed at any given value), imposing that the bottom row of G equal zero. There is a unique generalized inverse of $Ds - D\chi$ satisfying these two conditions. The conditions thus identify G.

Then, i's marginal "altruistic impact" of demand-shift dx^i, starting from a given WES, equals $\psi^i \cdot dx^i$, where

$$\psi^{iT} = \nabla w^i \cdot Ds\, G; \tag{26}$$

G converts the demand-shift to a price-shift, Ds converts the price-shift to a supply-shift, and ∇w^i converts the supply-shift to a marginal impact on i's altruistic utility. Transposing (26) yields (23).

Example

To illustrate the equilibrium concept, consider the economy $\mathcal{E}^{(n)}$, an arbitrary n-replication of the following economy $\mathcal{E}$ with $I = 2$ individuals and $L = 2$ goods:

$$\begin{aligned} e^i &= (3/2, 0)^T \;\; \forall i, \\ \theta^i &= 1/2 \;\; \forall i, \\ v^1(x^1) &= \ln(x_1^1) + x_1^1/3 + x_2^1, \\ v^2(x^2) &= \ln(x_1^2) + x_2^2; \\ w^1(s) &= s_1 + 2s_2, \\ w^2(s) &= 0; \end{aligned} \tag{27}$$

and a production function

$$y(p) = \left(1 - 2\sqrt{\frac{p_2}{p_1}},\; 4 - 2\sqrt{\frac{p_1}{p_2}}\right) \tag{28}$$

as long as $p_1/p_2 \in [1/4, 4]$ to ensure an interior solution. Supply is therefore

$$s(p) = \left(4 - 2\sqrt{\frac{p_2}{p_1}},\; 4 - 2\sqrt{\frac{p_1}{p_2}}\right), \tag{29}$$

as long as $p_1/p_2 \in [1/4, 4]$.

Let $\bar{p} \equiv (1,1)$, $\bar{\psi}^1 \equiv (-1/3, 0)$, and $\bar{\psi}^2 \equiv (0,0)$, and let

$$\bar{x}^i(p,\pi) \equiv \underset{x^i}{\operatorname{argmax}}\; \tilde{u}^i(x^i) \;\Big|\; p \cdot x^i \le p \cdot e^i + \theta^i \pi \;\; \forall i \tag{30}$$

$$\text{where} \quad \tilde{u}^i(x^i) \equiv v^i(x^i) + \bar{\psi}^i \cdot x^i = \ln(x_1^i) + x_2^i \;\; \forall i \tag{31}$$

$$\implies \bar{x}_1^i(p,\pi) = \min\left(\frac{3p_1 + \pi}{2p_1}, \frac{p_2}{p_1}\right) \;\; \forall i, \tag{32}$$

$$\bar{x}_2^i(p,\pi) = \max\left(0, \frac{3p_1 + \pi}{2p_2} - 1\right) \;\; \forall i. \tag{33}$$

Note that the specified impact of $\bar{\psi}_1^1 = -1/3$ cancels out the $+x_1^1/3$ term of (27), leaving the maximand $\tilde{u}^i(x^i)$ equal for both i. The individuals' demand functions are then identical because the individuals also have the same endowments and profit shares.

We will now show that $(\bar{p}, \{\bar{x}^i(\cdot)\}^{(n)})$ is a CESE of $\mathcal{E}^{(n)}$.

To take care of the preliminaries, first observe that demand function profile $\{\bar{x}^i(\cdot)\}$ is admissible in $\mathcal{E}$. As noted following Definition 9, it follows that $\{\bar{x}^i(\cdot)\}^{(n)}$ is admissible in $\mathcal{E}^{(n)}$. Observe also that $\{\bar{x}^i(\cdot)\}$, and thus $\{\bar{x}^i(\cdot)\}^{(n)}$, is locally $\mathcal{C}^1$ around $(\bar{p}, \pi)$ for any $\pi \geq 0$.

Next, let us confirm that $\bar{p}$ is a WES of $\mathcal{E}$ and demand function profile $\{\bar{x}^i(\cdot)\}$. From (28), $y(\bar{p}) = (-1, 2)$. Also, $\tilde{u}^i(x^i)$ is strictly increasing for both i, so both individuals exhaust their budgets at any p, π. Thus $\bar{\pi} \equiv \pi(\bar{p}, \bar{x}(\cdot)) = \bar{p} \cdot y(\bar{p}) = 1$. $\bar{x}(\bar{p}, \bar{\pi}) = s(\bar{p}) = (2, 2)$, confirming that $\bar{p}$ is a WES.

Then, by substituting $p \cdot y(p)$ for π in (33) and summing across i, we have aggregate implicit demand $\bar{\chi}(p)$. From here, though we will not work through the details, it is straightforward to find excess demand $\bar{z}(\cdot)$ $(\equiv \bar{\chi}(\cdot) - s(\cdot))$. With normalized excess demand $\hat{\bar{z}}(\cdot)$ defined as in Definition 4, we can then confirm that $\bar{p}$ is regular by Definition 6.

Finally, by Proposition 1, it follows that $\bar{p}$ is also an RWES of $(\mathcal{E}^{(n)}, \{\bar{x}^i(\cdot)\}^{(n)})$.

What makes the proposed $(\bar{p}, \{\bar{x}^i(\cdot)\})$ a CESE, beyond the fact that $\bar{p}$ is an RWES of $(\mathcal{E}, \{x^i(\cdot)\})$, is that $\bar{\psi} = \psi(\bar{p}, \bar{x}(\cdot))$. This can be calculated from the Definition (23) of $\psi(\cdot)$ (and the prior Definition 22 of $G(\cdot)$), but the identity may be understood as follows.

Let $\tilde{G}^{(n)}(p, x(\cdot))$ denote an $L \times L$ (here, 2×2) matrix mapping individual demand-changes into changes to equilibrium prices, in economy $\mathcal{E}^{(n)}$, given initial conditions in which aggregate demand is $x(\cdot)$ and the price vector p is an RWES of $(\mathcal{E}^{(n)}, x^{(n)}(\cdot))$.[9] Since excess demand is h.o.d. 0, we can without loss of generality fix p_L, i.e. require that $\tilde{G}(\cdot)$ have a bottom row of zeroes.

Now observe that $\bar{x}_2^i(\bar{p}, \bar{\pi}) > 0 \; \forall i$. The quasilinearity of (31) guarantees that, at $(\bar{p}, \bar{\pi})$, each i's marginal purchases are exclusively of good 2. Therefore

$$\delta(\bar{p}, \bar{x}(\cdot)) = (0, 1/\bar{p}_2)^T = (0, 1)^T. \tag{34}$$

This in turn implies that if one i reduces x_2^i, this simply increases profits, which others spend entirely on good 2. Supply levels do not change, and neither do prices. (A price-change would induce a supply-change.) In other words, $G(\bar{p}, \bar{x}(\cdot))(0, -1)^T = (0, 0)^T$. The upper-right entry of $G(\bar{p}, \bar{x}(\cdot))$ therefore equals zero.

[9]The tilde distinguishes $\tilde{G}^{(n)}(\cdot)$ from what would more naturally be denoted $G^{(n)}(\cdot)$—i.e. the analog of $G(\cdot)$ in economy $\mathcal{E}^{(n)}$—but which is independent of n. The latter maps marginal individual demand-changes dx^i to what marginal equilibrium price-changes *would* be if i also responded to the marginal price-change she herself induced. This distinction vanishes as n grows.

The only remaining unknown entry of $G(\bar{p}, \bar{x}(\cdot))$ is its upper-left, representing the extent to which marginal purchases of good 1 increase the price of good 1. To find it, we will consider a marginal individual demand-change proportional to $(1, -1)^T$. Since $\bar{p}_1 = \bar{p}_2$, this demand-change by i is orthogonal to $\bar{p}$.

By $s(p)$ from (29), $\partial s_1(p)/\partial p_1 = 1$ and $\partial s_2(p)/\partial p_1 = -1$. That is, recalling that supply in $\mathcal{E}^{(n)}$ equals $ns(p)$, each marginal unit increase in p_1 from the $\bar{p}$ baseline induces an n-unit increase in the supply of good 1 and an n-unit decrease in the supply of good 2 in economy $\mathcal{E}^{(n)}$.

Likewise, by (32)–(33), each marginal unit increase in p_1 induces a 1-unit decrease in demand for good 1 by all $2n-1$ consumers other than i. (Holding profits fixed, it also induces a 1-unit increase in demand for good 2. We have not shown that profits will in fact remain fixed as prices change; but we know that profit-changes will not affect demand for good 1, as long as all consumers are consuming a positive quantity of good 2.)

To maintain market clearing, p_1 must rise by just enough to induce an increase in s_1, and a decrease in x_1^{-i}, which sum to 1: the additional unit of good 1 which consumer i has resolved to buy. That is, we must have

$$n\, dp_1 + (2n-1)(dp_1) = 1$$
$$\implies dp_1 = \frac{1}{3n-1}.$$

As noted above, changes in demand for good 2, starting from the baseline of $(\bar{p}, \bar{x}(\cdot))$, have no impact on supply levels. It follows that $\psi_2^1(\bar{p}, \bar{x}(\cdot)) = \psi_2^2(\bar{p}, \bar{x}(\cdot)) = 0$, as desired. Moreover, since $w^2(s) = 0$, $\psi_1^2(\bar{p}, \bar{x}(\cdot)) = 0$ by (23), as desired. All that remains is to show that $\psi_1^1(\bar{p}, \bar{x}(\cdot)) = -1/3$.

Recall that the marginal individual demand-shift of $(1, -1)$ induces a marginal price-shift of $dp_1 = \frac{1}{3n-1}$. This price-shift, in turn, induces a marginal $\frac{n}{3n-1}$-unit increase in the equilibrium supply of good 1 and $\frac{n}{3n-1}$-unit decrease in the equilibrium supply of good 2. Recalling that $w^1(s) = s_1 + 2s_2$, the marginal altruistic impact of marginal individual demand-shift $(1, -1)$, from the perspective of consumer 1 (or any of her clones), equals $-\frac{n}{3n-1}$. As $n \to \infty$, this altruistic impact approaches $-1/3$. By definition, therefore, $\psi_1^1(\bar{p}, \bar{x}(\cdot)) - \psi_2^1(\bar{p}, \bar{x}(\cdot)) = -1/3$. But $\psi_2^1(\bar{p}, \bar{x}(\cdot)) = 0$. Thus $\psi_1^1(\bar{p}, \bar{x}(\cdot)) = -1/3$, as desired.

With $\bar{\psi} = \psi(\bar{p}, \bar{x}(\cdot))$, $(\bar{p}, \bar{x}(\cdot))$ is a CESE of $\mathcal{E}$ and any of its replications.[10]

As this example illustrates, the impacts of consumer behavior after accounting for general equilibrium effects can differ substantially from the impacts one finds when

[10] $(\bar{p}, \bar{\psi})$ is also an RCESE of $\mathcal{E}$ and its replications, where RCESE is a refinement of CESE defined in Section 3.2 below. Furthermore, though we have not shown this here, this RCESE is unique.

one entirely ignores substitution by other parties. Inspecting $w^1(s)$ alone, one might expect that consumer 1 assigns an altruistic impact of 1 to purchasing a unit of good 1, and an altruistic impact of 2 to purchasing a unit of good 2, on any margin. Here, by contrast, we find that she assigns a negative weight to good 1 and a weight of 0 to good 2.

Likewise, general equilibrium impacts can differ substantially from those found in partial equilibrium. Expressions (29) and (32) record an upward-sloping supply curve and a downward-sloping demand curve for both goods, respectively, around $p = (1, 1)$. A partial equilibrium analysis, like that of Kaufmann et al. (2024), would therefore conclude that, from individual 1's perspective, the altruistic impact of purchasing a unit of good 1 given prices $(1, 1)$ lay in $(0, 1)$, and that of purchasing a unit of good 2 lay in $(0, 2)$.

Discussion

Proposition 3 states that each consumer i is indifferent between all demand functions which demand, at the equilibrium price and profit level, the basket x^i that maximizes $v^i(x^i) + \psi^i(\bar{p}, \bar{x}(\cdot)) \cdot x^i$ subject to her budget constraint. The proposition thus offers little guidance as to what sorts of behavior we might expect to see in equilibrium. The (cross-)price elasticities of demand i chooses *around* the equilibrium price and profit level are of no consequence for i, but because they affect $x(\cdot)$ and thus ψ^j for $j \neq i$, they affect other consumers' best-response demand functions. That is, in a CESE, consumers' demands are highly sensitive to their fellow consumers' arbitrary choices of threatened out-of-equilibrium behavior. Exotic behavior may therefore be motivated in equilibrium by mutually best-responding threats with no basis in anyone's preferences over supply or own consumption. A refinement of CESE, designed to rule out these "non-credible threats", will be discussed in Section 3.2.

Even in the absence of such a refinement, however, Proposition 3 offers complete guidance as to what *baskets* an individual in a large economy, with given preferences and a given endowment, should be content to buy at given prices and profits. Again, the individual should buy a basket x^i that maximizes $v^i(x^i) + \psi^i(\bar{p}, \bar{x}(\cdot)) \cdot x^i$ subject to her budget constraint. If our goal is only to advise informed consumers with given altruistic preferences, therefore, our analysis of equilibrium can end here. Further analysis of the impacts of altruistic concerns on a consumer's behavior may be found in Section 4.

3.2 Refinement

Let us refer to ψ^i as i's <u>impact vector</u>. Let ψ denote the $L \times I$ <u>impact matrix</u> whose column i equals ψ^i.

Given ψ^i, let us refer to

$$\tilde{u}^i_{[\psi^i]}(x^i) \equiv v^i(x^i) + \psi^i \cdot x^i$$

as i's quasi-utility function. $\tilde{u}^i_{[\psi^i]}(\cdot)$, for any $\psi^i \in \mathbb{R}^L$, represents preferences over all-things-considered preferences over purchasing choices $x^i \in \mathbb{R}^L_+$. These preferences incorporate both the private benefits of consuming a given basket and the altruistic impacts of the supply-changes induced by purchasing that basket.

As discussed at the end of Section 3.1, we would like a refinement of CESE in which individuals choose demand functions that behave reasonably not only at the precise $(\bar{p}, \bar{\pi})$ that obtain in economic equilibrium but at all (p, π), or at least all (p, π) near $(\bar{p}, \bar{\pi})$. Such demand functions can be motivated by giving each individual uncertainty about the prices and profits she will face.

Unfortunately, how an individual best responds to a shock to prices and profits depends on the source of the shock. This is because the impacts of a given consumption decision depend on the economic conditions determining prices and profits.

Impacts are approximately independent of the shocks to the prices and profits an individual faces if these shocks do not significantly affect supply or others' budget sets across the economy at large. They are approximately independent, for instance, if the shocks consist of idiosyncratic costs that the individual faces the way to the shop, or that his local shop faces in stocking a good on a particular day. He then does best by choosing x^i to maximize $\tilde{u}^i_{[\psi^i]}$ subject to a budget constraint that varies with p and π.

If the shock to (p, π) stems from a shock to production, on the other hand, our individual can anticipate that the supply vector now lies at a location in his altruistic utility function $w^i(\cdot)$ different from the location that would have been realized absent the shock, and this may alter ∇w^i. The shock to supply may also shift the value of Ds. Finally, others' demands will adjust for the same reasons as his own, potentially shifting $D\chi$. For all three reasons, in the event of a shock, the individual does best to demand the basket that maximizes some

$$v^i(x^i) + \psi^i(p, \pi) \cdot x^i,$$

where not only the budget constraint but also the impact vector ψ^i varies with p and π.

Finally, if the shock to (p, π) stems from a shock to the tastes of individuals other than i, the appropriate adjustment to ψ^i depends on the details of the shock. Without knowing the distribution of tastes and individuals from which the shocks are drawn—and certainly without knowing whether the observed (p, π) shocks stem from taste or supply shocks—observing dp and $d\pi$ is not enough to determine the appropriate adjustment, if any, to ψ^i. A complete treatment of out-of-equilibrium demands would require individuals to choose demand functions that depend not only on prices and profits but also, directly, on a third variable indexing the realization of the shock to production and/or tastes.[11]

[11] Kaufmann et al. (2024) avoid these difficulties by imposing that individuals choose demand

This reveals the difficulty of defining reasonable behavior by altruistic consumers in the event that prices or profits are out of equilibrium (absent strong linearity assumptions, as detailed in footnote 11). For simplicity, and perhaps (not unrelatedly) for realism, we will choose the refinement in which individuals understand altruistic impacts to be independent of price and profit shocks.

Definition 12. *A robust CESE (RCESE) is a CESE* $(\bar{p}, \{x^i(\cdot)\})$ *in which each* $x^i(\cdot)$ *maximizes* $\tilde{u}^i_{[\psi^i]}$ *at all* $p \gg 0, \pi \geq 0$*.*

Note that, if $w^i(\cdot) = 0$, $\psi^i(\cdot) = 0$ as well. One desirable feature of RCESE is therefore that individuals without altruistic preferences adopt the demand functions that globally maximize $v^i(\cdot)$, as individuals are assumed to do in the conventional Walrasian setting. CESE, by contrast, requires only that an individual with $w^i(\cdot) = 0$ adopt a demand function that maximizes $v^i(\cdot)$ at equilibrium prices and profits.

RCESE is therefore a relatively straightforward generalization of Walrasian equilibrium, after technicalities around regularity, to the case in which individuals have preferences over total supply levels. This will also be stated formally in Proposition 4, in the section below.

After this subsection, we will restrict our attention to RCESE. Under some conditions, this lets us reframe the model in simpler terms.

Instead of positing that each i chooses an arbitrary admissible demand function $x^i(\cdot)$, we can posit that i chooses a vector ψ^i. If, given ψ, a unique demand function for each i maximizes $\tilde{u}^i_{[\psi^i]}(x^i)$, we may define

$$x^i_{[\psi^i]}(p, \pi) \equiv \underset{x^i}{\text{argmax}} \ \tilde{u}^i_{[\psi^i]}(x^i) \ \Big| \ p \cdot x^i \leq p \cdot e^i + \theta^i \pi,$$ [12]

and let $x_{[\psi]}(\cdot)$ denote the sum of individual $x^i_{[\psi^i]}(\cdot)$. With

$$\pi(p, \psi) \equiv \pi(p, x_{[\psi]}(\cdot)), \tag{35}$$

we can then define the implicit individual demand function compatible with a given ψ as

$$\chi^i_{[\psi]}(p) \equiv \underset{x^i}{\text{argmax}} \ \tilde{u}^i_{[\psi^i]}(x^i) \ \Big| \ p \cdot x^i \leq p \cdot e^i + \theta^i \pi(p, \psi),$$

functions that not only are locally $\mathcal{C}^1$—i.e. have locally continuous derivatives—but have locally *constant* derivatives; and by assuming a supply function and symmetric altruistic utility functions such that, in our notation, Ds globally constant and Dw^i is globally constant for all i. They then observe a sense in which, under these conditions, optimal demand-shifts given supply shocks are approximately linear in the induced price-shifts, as desired.

[12] Observe that $x^i_{[\psi^i]}(\cdot)$ is admissible wherever it is defined.

and let

$$\chi_{[\psi]}(p) \equiv \sum_{i=1}^{I} \chi^i_{[\psi]}(p),$$
$$z_{[\psi]}(p) \equiv \chi_{[\psi]}(p) - s(p).$$

As with $\pi(\cdot)$ in (35), we can define

$$\delta(p, \psi) \equiv \delta(p, x_{[\psi]}(\cdot)),$$

and analogously extend (23) and the preceding terms—defined there as functions of admissible aggregate demand functions $x(\cdot)$—to be defined with impact matrices, and not only explicit aggregate demand functions, in their second arguments. Finally, we can define (p, ψ) to be an RCESE if $(p, \{x^i_{[\psi^i]}(\cdot)\})$ is an RCESE:

Definition 13. *Given $\bar{\psi}$, let $\bar{p}$ be an RWES of an economy $\mathcal{E}$ and a demand function profile $\{x^i_{[\bar{\psi}^i]}(\cdot)\}$ that is [defined and] locally regular around $(\bar{p}, \bar{\pi})$. Then $(\bar{p}, \bar{\psi})$ is an RCESE of $\mathcal{E}$ if*

$$\psi(\bar{p}, \bar{\psi}) = \bar{\psi}.$$

3.3 Existence

As noted in Section 3.2, if there are no altruists, then RCESE is essentially equivalent to Walrasian equilibrium, subject to certain technicalities. We can state this result more formally:

Proposition 4. *RCESE generalizes regular Walrasian equilibrium*
Let $\bar{p}$ be a Walrasian equilibrium of an economy $\mathcal{E}$ with $w^i(\cdot) = 0\ \forall i$. If $\bar{p}$ is regular and $\{x^i_{[0]}(\cdot)\}$ is locally regular around $(\bar{p}, \bar{p} \cdot y(\bar{p}))$, then $(\bar{p}, \mathbb{0}_{L\times I})$ is an RCESE of $\mathcal{E}$.

Proof. Because $\bar{p}$ is a regular Walrasian equilibrium, $\bar{p}$ is an RWES (with $\bar{\pi} = \bar{p} \cdot y(\bar{p})$). Because each $w^i(\cdot) = 0$, each $\psi^i(\cdot) = \mathbb{0}_L$, by (23). Thus $\psi(\bar{p}, \mathbb{0}_{L\times I}) = \mathbb{0}_{L\times I}$. □

Determining when an RCESE exists in a less trivial setting is difficult, but existence can be proven under certain conditions.

Let Δ^{L-1} denote the simplex in $\mathbb{R}^L$. Given a price vector p, let

$$\tilde{p} \equiv p/(\mathbb{1}_L \cdot p) \in \Delta^{L-1}$$

denote its simplex normalization.

For a set A, let A° denote its interior and A^c denote its boundary.

Definition 14. *A supply function $s(\cdot)$ is* <u>*uniformly smooth*</u> *on a compact, convex set of price vectors $P \subset \Delta^{(L-1)\circ}$ if*

1. *$s(p) \gg 0$ iff $\tilde{p} \in P^\circ$ and*

2. *$s(\cdot)$ is $\mathcal{C}^1$ on P with $D\hat{s}(\hat{p})$ nonsingular for all $\hat{p}$ with $\tilde{\check{\hat{p}}} \in P^\circ$,*

where the interior and boundary of P are defined with respect to the space Δ^{L-1}.

For intuition, in two goods, the supply possibility set corresponding to a uniformly smooth supply function must resemble Figure 2.

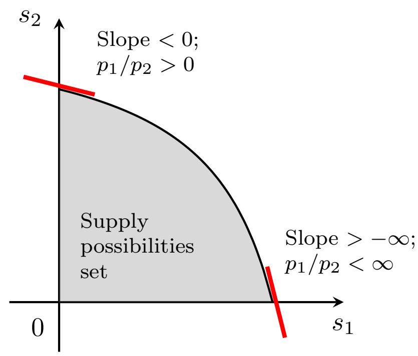

Fig. 2: Uniformly smooth supply in two goods

The relative price of good 1 p_1/p_2 motivating supply at a point s on the frontier is the negative of the slope of the tangent of the frontier at s (which is defined, by (iii)). The strict concavity of the frontier ensures that profit-maximizing production is unique at all prices. The compact range of slopes from the upper left to the bottom right ensures that, within this range of prices, supply is interior, and at or beyond its ends, the supply of some good is zero.

For $\mathcal{L} \subset \{1, ..., L\}$, let $\Delta_{\mathcal{L}}^{L-1}$ denote the face of Δ^{L-1} with entries of zero outside $\mathcal{L}$.

Definition 15. *A supply function $s(\cdot)$ is* <u>*quasi-monotonic*</u> *on a set of price vectors P if, for any $\mathcal{L} \subset \{1, ..., L\}$ and $d \in \Delta_{\mathcal{L}}^{L-1}$,*

$$p \in \underset{p' \in P}{\text{argmax}}\ p' \cdot d \implies s_\ell(p) > 0 \text{ for some } \ell \in \mathcal{L}.$$

That is, quasi-monotonicity on P ensures that if within P prices p maximize a weighted average of the prices of some set of goods $\mathcal{L}$, then the supply of at least one good in $\mathcal{L}$ is positive.

Proposition 5. ***Existence of RCESE given uniform smoothness and quasi-monotonicity of supply, separability of consumption utility, and symmetric consumers***

Suppose that $s(\cdot)$ is uniformly smooth and quasi-monotonic on a compact, convex set of price vectors $P \subset \Delta^{(L-1)\circ}$. Suppose also that individuals have symmetric utility functions $u^i(\cdot) = u(\cdot)$, endowments $e^i = e/I$, and profit shares $\theta^i = 1/I$; and that consumption utility is additively separable,

$$v(x^i) = \sum_{\ell=1}^{L} v_\ell(x^i_\ell), \tag{36}$$

with $v_\ell(\cdot)$ $\mathcal{C}^2$, strictly increasing, and satisfying

$$v^{i\prime\prime}_\ell(x^i_\ell) < 0, \quad \lim_{x^i_\ell \to 0} v^{i\prime}_\ell(x^i_\ell) = \infty \ \ \forall \ell.$$

Then an RCESE $(\bar{p}, \bar{\psi})$ exists, with $\tilde{\bar{p}} \in P^\circ$, and $\bar{\psi}^i$ and x^i the same for all i.

Proof. See Appendix B.4. □

Proposition 6. ***Existence of RCESE given uniform smoothness of supply and separability and quasilinearity of consumption utility***
Suppose that:

1. *$s(\cdot)$ is uniformly smooth on a compact, convex set of price vectors $P \subset \Delta^{(L-1)\circ}$.*
2. *Either*
   (a) *$s(\cdot)$ is quasi-monotonic on P or*
   (b) *for all ℓ, $\exists i : w^i(\cdot) = 0, e^i_\ell > 0$.*
3. *For all i, $v^i(\cdot)$ is additively separable and quasilinear in good L,*

$$v^i(x^i) = \sum_{\ell=1}^{L-1} v^i_\ell(x^i_\ell) + v^i_L x^i_L, \tag{37}$$

with $v^i_\ell(\cdot)$ $\mathcal{C}^2$, strictly increasing, and satisfying $v''_\ell(x^i_\ell) < 0$ and $\lim_{x^i_\ell \to 0} v'_\ell(x^i_\ell) = \infty$ for all $\ell < L$.

Then for each i there is a $\underline{B}^i_L > 0$ and a $\underline{v}^i_L \geq 0$ such that if, for all i,

1. *$v^i_L > \underline{v}^i_L$ and*
2. *$p \cdot e^i_L + \theta^i p \cdot y(p) \geq \underline{B}^i_L \ \forall p \in P$,*

then an RCESE $(\bar{p}, \bar{\psi})$ exists, with $\bar{\psi}^i_L = 0 \ \forall i$, $\bar{\psi}^i_\ell < v^i_L \ \forall i$, and $\tilde{\bar{p}} \in P^\circ$.

Proof. See Appendix B.5. □

Existence more generally

The conditions required for Propositions 4–6 are highly restrictive because it is difficult to create economic conditions that guarantee the existence of an RWES throughout a large space of potential values of ψ. Well-known results establish that economies "generically" have regular and indeed only regular Walrasian equilibria, in the sense that singular Jacobians of excess demand around equilibrium price vectors can be made nonsingular with arbitrary perturbations to the economic primitives (see e.g. Mas-Colell et al. (1995), Proposition 17.D.5). In the current setting, however, genericity results of this kind are not obviously useful, because ψ is endogenous to prices and economic primitives, and may in principle adjust to any perturbations so as to push equilibria toward regularity.

Despite this obstacle to identifying flexible conditions guaranteeing the existence of RCESE, it can easily be shown that RCESE exist far beyond the narrow conditions above. Given an economy $\mathcal{E} = (E, \{u^i(x^i)\})$, let $\bar{p}$ be an RWES of $(E, \{x^i_{[0]}(\cdot)\})$. Let $\psi = \psi(\bar{p}, \mathbb{0}_{L\times I})$, so that ψ is the best-response impact matrix at $\bar{p}$ to $\psi = \mathbb{0}_{L\times I}$. Finally, let

$$\mathcal{E}(\psi) \equiv (E, \{\tilde{v}^i_{[\psi^i]}(\cdot) + w^i(\cdot)\}),$$

where

$$\tilde{v}^i_{[\psi^i]}(x^i) \equiv v^i(x^i) - \psi^i \cdot x^i. \tag{38}$$

That is, let $\mathcal{E}(\psi)$ denote the economy with the same environment as $\mathcal{E}$, and whose individuals have the same altruistic utility functions $\{w^i(\cdot)\}$ as in $\mathcal{E}$, but who derive ψ^i_ℓ fewer units of consumption utility from good ℓ at all margins. Then, defining $\{x^i_{[\cdot]}(\cdot)\}$ with respect to the modified consumption utility functions, so that

$$x^i_{[0]}(p, \pi) \equiv \underset{x^i}{\operatorname{argmax}}\ \tilde{v}^i_{[\psi^i]}(x^i) \ \Big|\ p \cdot x^i \leq p \cdot e^i + \theta^i y(p),$$

$(\bar{p}, \psi)$ is an RCESE of $(\mathcal{E}(\psi), \{x^i_{[0]}(\cdot)\})$. This is because if each i's consumption utility is given by $v^i(x^i) - \psi^i \cdot x^i$ and she adopts impact vector ψ^i, then her quasi-utility function is simply $v^i(\cdot)$: just as it is if her consumption utility is $v^i(\cdot)$ and she adopts impact vector 0. The best response impact matrix in the first case thus equals ψ, just as in the second.[13,14]

Multiplicity

Under standard assumptions, including ours, if consumers are symmetric or have quasilinear utility functions then there is a unique normalized Walrasian equilibrium.

[13] The example of Section 3.1 can be constructed in this way, beginning from the economy in which $v^i(\cdot) = \ln(x^i_1) + x^i_2 \ \forall i$.

[14] If $\psi^i_\ell < 0$ for some i, ℓ, $\tilde{v}_{[\psi^i]}(\cdot)$ as defined by (38) may decrease in x^i_ℓ on some margins even if $v^i(\cdot)$ is increasing. As shown in the proof of Proposition 9 (Appendix B.6), a profile of consumption utility functions compatible with RCESE $(\bar{p}, \bar{\psi})$ that preserves monotonicity can also be constructed.

One might therefore wonder whether uniqueness extends to RCESE in the context of Propositions 5 and 6.

In general, it does not. Kaufmann et al. (2024) study a narrower setting in which consumers are symmetric *and* have well-behaved quasilinear consumption utility functions.[15] Indeed they also assume that there are only two goods (let us stipulate that the second is the quasilinear good); that the supply of the first increases linearly in its relative price; and that, after translating notation, the common altruistic utility function $w(\cdot)$ takes the form $w(s) = -\kappa s_1$, so that it is independent of the quasilinear good and linearly decreases in the other. These restrictions ensure that the relationship between the altruistic utility function and the altruistic impacts is well-behaved: in particular, in any RCESE, $\psi_2^i = 0$ and $\psi_1^i \equiv \psi_1 \in (-\kappa, 0]$ is the same for all i. Even so, however, Kaufmann et al. find that there may be multiple RCESE.

For intuition, higher absolute values of ψ_1 motivate individuals to consume less of good 1. If they strongly prefer to consume at least a small quantity of good 1, their demand may be less responsive to price-changes once their consumption of good 1 has fallen. Lower price-responsiveness of demand (in combination with the constant price-responsiveness of supply) then implies that if a given individual chooses to consume more of good 1, the impact on the supply of good 1 is greater. Both low and high absolute values of ψ_1 may therefore be self-confirming.

Kaufmann et al. also study the welfare implications of this multiplicity in their setting. In particular they note that equilibria in which $|\psi_1|$ is higher, and thus the consumption of good 1 is lower, Pareto-dominate those in which $|\psi_1|$ is lower. An implication is that the first fundamental theorem of welfare economics does not generally hold when individuals have preferences over supply, even if these preferences are identical and motivate consumers to shift their demands to some extent in the collectively desired direction. This is unsurprising in light of the close analogy between the game of strategic demand choice, in which individuals have preferences over supply, and a traditional public good game.

We will not explore uniqueness further here. The null uniqueness results cited above suffice to show that assumptions guaranteeing the uniqueness of Walrasian equilibrium do not guarantee the uniqueness of equilibrium in the current setting; and the following section will demonstrate that essentially no other general statements on the implications of altruism can be made in the current setting.

[15] Strictly speaking, because they allow the supply and consumption of goods to be negative, their setting is not narrower than those considered here. Their multiplicity result however does not rely on this generalization.

4 Implications for consumer behavior

An unfortunate feature of the results of Section 3 is that the informational requirements for computing ψ^i are very demanding. To decide what to buy, an altruist i is asked to know the gradient of the aggregate Engel curve and the Jacobians of supply and implicit demand with respect to price (or, equivalently, the cross-price elasticity matrices of supply and implicit demand and the aggregate supply levels). One might therefore wonder whether it is possible to draw at least some incomplete conclusions about i's impact vector or optimal consumption basket directly from his preferences, without relying on much or any economic data.

Section 4.1 notes a narrow case in which i can easily infer ψ^i from $w^i(\cdot)$: $\psi^i = 0$ if (and only if) $\nabla w^i(s) \propto p$. Section 4.2, however, shows that essentially nothing else can be known about the relationship between $w^i(\cdot)$ and ψ^i from first principles. Section 4.3 details an implication of this point: that even if $w^i(\cdot)$ increases in every good, ψ^i may be strictly negative, and indeed so negative that i prefers not to exhaust his budget.

4.1 Prices and altruistic preferences

Proposition 7. ***Impacts zero if altruistic preferences proportional to prices***
Let (p, ψ) be an RCESE.

a. If $\nabla w^i(s(p)) \propto p$, then $\psi^i = 0$.

b. If $\nabla w^i(s(p)) \not\propto p$ and $\text{Rank}(Ds(p)) = L - 1$, then $\psi^i \neq 0$.

Proof. Part (a) follows from definition (23) modified for the RCESE context—

$$\psi^i(p, \psi) \equiv (Ds(p)G(p, \psi))^T \nabla w^i(s(p)) \tag{39}$$

—and the fact that $p \cdot Ds(p) = \mathbb{0}_L^T \ \forall p$.

Part (b) follows from (39) and the fact that, if (p, ψ) is an RCESE, then $-Dz_{[\psi]}(p)$ and so also its generalized inverse $G(p, \psi)$ are of rank $L - 1$. Since the bottom row of $G(p, \psi)$ consists of zeroes, its left null space $\text{Null}(G(p, \psi)^T)$ is spanned by the unit row vector with a one in place L. The column space of $Ds(p)^T$ is orthogonal to its left null space, which is spanned by p. So if $\text{Rank}(Ds(p)) = L - 1$ and $\nabla w^i(s(p)) \not\propto p$, then $Ds(p)^T \nabla w^i(s(p)) \perp p$, so $Ds(p)^T \nabla w^i(s(p)) \notin \text{Null}(G(p, \psi)^T)$. □

In Proposition 4 and its proof, we noted trivially that if a consumer i is indifferent to supply levels ($w^i(\cdot) = 0$), he is also indifferent to potential purchases' impacts on supply levels ($\psi^i = 0$). Part (a) shows that this antecedent is stronger than necessary. A consumer may feel that the production of each good has desirable or undesirable consequences, but as long as he feels the significance of these consequences to be proportional on the current margin to the goods' prices, he assigns no altruistic impact to any marginal purchases. He may support policies that would increase or decrease aggregate productive capacity, but he is indifferent to supply-shifts of the only sort that consumer decisions (at least in this model) can achieve: namely shifts within the hyperplane that contains $s(p)$ and is orthogonal to p.

Part (b) then shows that the $\nabla w^i(s(p)) \propto p$ condition cannot in general be weakened further. If a consumer feels that the the production of goods has desirable or undesirable consequences not proportional on the current margin to the goods' prices, then unless there is a direction in which relative prices can shift without affecting supply levels at all, he does put a nonzero altruistic impact on the consumption of some good.

Part (a) tells us that p is in the null space of $(Ds(p)G(p,\psi))^T$. This result has a dual: p is *not* in the column space.

Proposition 8. ***Impacts zero if proportional to prices***
Let (p, ψ) be an RCESE. Then $\psi^i \propto p$ (if and) only if $\psi^i = 0$.

Proof. By construction, $\delta(p,\psi) \cdot G(p,\psi)^T = 0$. But $\delta(p,\psi) \not\perp p$, since by definition of RCESE $\frac{\partial}{\partial \pi}(p \cdot x_{[\psi]}(p,\pi)) = p \cdot \delta(p,\psi) > 0$. If $\psi^i(p,\psi) \propto p$ but $\neq 0$, left-multiplying both sides of (39) by $\delta(p,\psi)$ yields $\delta(p,\psi) \cdot \psi^i(p,\psi) = 0$, a contradiction. □

For interpretation, suppose $\psi^i \propto p$. Then any demand-shift dx^i that keeps i on her budget constraint ($dx^i \perp p$) has on balance no altruistic impact ($\psi^i \cdot dx^i = 0$). Any dx^i can be decomposed into a weighted sum of a vector orthogonal to p and a vector proportional to δ, which changes her total expenditure but does not affect prices or supply. So if there is no altruistic impact associated with a demand-shift $dx^i \perp p$, there is no altruistic impact associated with any demand-shift.

Observe that, as long as $\chi^i_{[\psi]}(p)$ is interior and $v^i(\chi^i_{[\psi]}(p))$ is locally differentiable with respect to χ^i, the optimality of i's demand requires

$$\nabla_{\chi^i} v^i(\chi^i_{[\psi]}(p)) + \psi^i \propto p.$$

An implication of Proposition 8, therefore, is that if i does assign some altruistic impact to the consumption of any good—e.g. if the conditions of Proposition 7b hold—then this in turn must somehow affect her consumption decisions, given the interiority and differentiability conditions.

4.2 Anything goes

Toward the end of Section 3, under "Existence more generally", we briefly observed how economic primitives, in particular consumption utility functions, can be chosen so that the economy admits a given impact matrix in equilibrium. This section extends that observation. For *any* relationship between altruistic preferences $w^i(\cdot)$ and impacts ψ^i except those ruled out in Section 4.1, a well-behaved economy can be constructed to match it.

Recall the Sonnenschein-Mantel-Debreu theorem. The following is a weakening of it sufficient to suggest the application to our case:

Given a price vector p and an $L\times L$ matrix A satisfying $Ap = \mathbb{0}_L$ and $p\cdot A = \mathbb{0}_L^T$, there is an economy $\mathcal{E}$, with $v^i(\cdot)$ nondecreasing, locally nonsatiated, and strictly concave for all i, such that $z_{[\mathbb{0}_{L\times I}]}(p) = 0$ and $Dz_{[\mathbb{0}_{L\times I}]}(p) = A$.

In short, price-changes may induce arbitrary changes to excess demand around an equilibrium price vector.[16]

In the present context, the theorem has a dispiriting converse. Changes to excess demand—in particular, the changes an individual might make to her own demand function, holding others' fixed—may induce arbitrary changes to equilibrium prices, and thus to quantities supplied. It then follows that as long as prices p are not proportional to $\nabla w^i(s(p))$, the only constraint we can impose from first principles on i's equilibrium impact vector is that it too is not proportional to p (see Propositions 7a and 8).

Proposition 9. ***Anything goes***
Choose

1. *a profile of I $\mathcal{C}^1$ functions $\{w^i(\cdot)\}$ from $\mathbb{R}^L_{++}$ to $\mathbb{R}$, with $I \geq 2L+1$ and $L \geq 2$;*
2. *an L-vector $s \gg 0$;*
3. *an L-vector $p \gg 0$; and*
4. *an $L \times L$ matrix M with $p \in \text{Null}(M)$ and $p \notin \text{Col}(M)$,*

and define ψ by $\psi^i = M\nabla w^i(s)\ \forall i$.

Then there is an environment E and a profile of increasing, strictly concave consumption utility functions $\{v^i(\cdot)\}$ such that $s(p) = s$ and (p, ψ) is an RCESE of economy $(E, \{v^i(\cdot) + w^i(\cdot)\})$.

Proof. See Appendix B.6. □

The example of Section 3.1 hinted at Proposition 9, in that it featured an equilibrium impact vector with non-positive entries ($\psi^i = (-1/3, 0)^T$) for an individual i with $\nabla w^i \gg 0$. To illustrate the the strength of the proposition more fully, consider

[16] The theorem is therefore sometimes known as the "anything goes" theorem. The weakened statement of it above is adapted from Mas-Colell et al. (1995), Proposition 17.E.2. and Mantel (1974).

the case in which $w^i(\cdot) \equiv w(\cdot)$ is the same for all i and $M\nabla w(s) = -\nabla w(s)$. Suppose for example that $p = (1,1)^T$, $\nabla w(s) = (1,-2)^T$, and

$$M = \frac{1}{3}\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix},$$

so that $\psi^i = (-1,2)^T\ \forall i$. Conditions can be constructed under which (p,ψ) is an RCESE. Though every individual agrees that it would be preferable to produce more of good 1 and less of good 2, every individual's altruistic considerations weigh against buying the former and in favor of buying the latter.

4.3 Satiation

To illustrate the strength of Proposition 9 perhaps even more fully, we will now construct an economy and an RCESE in which, for some i, $\nabla w^i \gg 0$ but $\psi^i \ll 0$—and in which the negative altruistic impact for i of any marginal purchase is large enough that, though $\nabla v^i \gg 0$ as well, i prefers not to exhaust her budget. We will then discuss what drives this result.

Example: "no ethical consumption under capitalism"

Consider the following economy $\mathcal{E}$ with $I = 2$ individuals and $L = 2$ goods:

$$\begin{aligned}
e^1 &= (1,1)^T \quad \forall i;\\
\theta^1 &= 0,\ \theta^2 = 1;\\
v^1(x^1) &= \frac{4}{3}\ln(x_1^1) - \frac{2}{15}\exp(-x_2^1), && (40)\\
v^2(x^2) &= \ln\left(x_1^2 - \frac{1}{2}\right) - 2\ln(3 - x_2^2) \text{ around } x^2 = (1,2)^T, && (41)\\
&\qquad \text{locally nonsatiated and globally strictly quasiconcave;}\\
w^1(s) &= s_1 + 2s_2,\\
w^2(s) &= 0;
\end{aligned}$$

and whose production possibilities frontier is given by

$$y_2 = 2 - (1+y_1)^2/4,$$

so that

$$s_1(p) = 2p_1/p_2, \quad s_2(p) = 3 - (p_1/p_2)^2. \tag{42}$$

We will show that $(\bar{p},\bar{\psi})$ with $\bar{p} = (1,1)^T$, $\bar{\psi}^1 = (-4/3,-2/15)^T$, $\bar{\psi}^2 = 0$ is an RCESE of $\mathcal{E}$, with an associated profit level $\bar{\pi} = 1$. At this RCESE, individual 1 will be seen to spend only half her budget.

Given (40), the value of x^1 that globally maximizes $\tilde{u}^1_{[\bar{\psi}^1]}(x^1)$ is $(1,0)^T$. This basket is more than affordable for individual 1 at $(\bar{p},\bar{\pi})$: its cost equals 1 and individual 1's budget equals 2. So $x^1_{[\bar{\psi}^1]}(p,\pi) = (1,0)^T$ around $(\bar{p},\bar{\pi})$, locally independent of p and (because $\theta^1 = 0$) globally independent of π.

Given (42), $s(\bar{p}) = e = (2,2)^T$ and

$$Ds(\bar{p}) = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}. \tag{43}$$

Given (41), $x^2_{[\mathbb{0}_2]}(p,\pi)$ is globally defined, by strict quasiconcavity; $x^2_{[\mathbb{0}_2]}(\bar{p},\bar{\pi}) = (1,2)^T$; and, fixing $p_2 = 1$, so that 2's budget equals $p_1 + 1 + \pi$,

$$x^2_{1[\mathbb{0}_2]}(p,\pi) = \frac{2-\pi}{p_1}, \qquad x^2_{2[\mathbb{0}_2]}(p,\pi) = p_1 + 2\pi - 1 \tag{44}$$

around $(\bar{p},\bar{\pi})$. Good 1 is thus locally inferior, and good 2 is locally normal, for individual 2. Recalling that $x^1(\cdot)$ is independent of π,

$$\delta(\bar{p},\bar{\psi}) = (1,2)^T. \tag{45}$$

Since $p \cdot x^2_{[\mathbb{0}_2]}(p,\pi)$ strictly increases without bound in π, by local nonsatiation, $\pi(p,\{x_{[\bar{\psi}]}(\cdot)\})$—the unique value of π setting

$$\bar{p} \cdot x_{[\bar{\psi}]}(\bar{p},\pi) = p \cdot s(\bar{p})$$

—is defined for all p. Around $\bar{p}$, still fixing $p_2 = 1$, we have

$$\pi(p,\{x_{[\bar{\psi}]}(\cdot)\}) = \frac{2p_1^2 - 6p_1 + 5}{2 - p_1}. \tag{46}$$

Substituting 1 for p_1, we have $\bar{\pi} \equiv \pi(\bar{p},\{x_{[\bar{\psi}]}(\cdot)\}) = 1$.

Substituting (46) for π into (44) gives

$$\chi^2_{1[\bar{\psi}]}(p) = \frac{2p_1^2 - 4p_1 + 1}{p_1^2 - 2p_1}, \qquad \chi^2_{2[\bar{\psi}]}(p) = \frac{3p_1^2 - 9p_1 - 8}{2 - p_1}. \tag{47}$$

Substituting 1 for p_1, we have $\chi^2_{[\bar{\psi}]}(\bar{p}) = (1,2)^T$. Since aggregate demand $(1,0)+(1,2)$ equals supply $(2,2)$, this confirms that $\bar{p}$ is a WES of $(\mathcal{E}, x_{[\bar{\psi}]}(\cdot))$.

Differentiating (47) with respect to p_1 and using the fact that $\chi^2(\cdot)$ must be h.o.d. 0 gives

$$D\chi^2_{[\bar{\psi}]}(\bar{p}) = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}. \tag{48}$$

Then (trivially) adding (48) to $D\chi^1_{[\bar{\psi}]}(\bar{p}) = \mathbb{0}_{2\times 2}$ to get $D\chi_{[\bar{\psi}]}(\bar{p})$; subtracting this from (43) to get $-Dz_{[\bar{\psi}]}(\bar{p})$; taking the appropriate generalized inverse compatible with

(45) to get $G(\bar{p}, \bar{\psi})$; and left-multiplying by (43), transposing, and right-multiplying by $\nabla w^1 = (1,2)^T$ confirms that $\psi(\bar{p}, \bar{\psi}) = \bar{\psi}$.

Four features of the economic setup drive the result above. Around the proposed equilibrium,

i. Individual 1 prefers to shift production from good 1 to good 2.

ii. Good 1 is inferior for individual 2.

iii. For individual 1, the marginal consumption utility of good 1 is significantly higher than the marginal consumption utility of good 2.

iv. For individual 1, the marginal consumption utility of each good is low, relative to the altruistic utility achievable by shifting supply.

By (i) and (ii), individual 1 shifts supply in a preferred direction when she decreases her spending. This effectively transfers income to individual 2, which he uses to shift supply toward good 2.

Individual 1 can increase her spending without impacting supply. However, this requires increasing her spending along the gradient $\bar{\delta}$: buying one fewer unit of good 1 and two more units of good 2 with each additional dollar spent instead of transferred to 2, so that the individuals' consumption baskets shift in opposite directions. By (iii), individual 1 prefers not to increase her spending in this direction. More generally, any spending increase by individual 1 with a weakly desirable supply impact would require a large enough decrease to x_1^1 per unit of increase to x_2^1 as to lower her consumption utility.

Finally by (iv), the negative altruistic impact associated with any spending increase by individual 1 that would increase her consumption utility is high enough that no spending increase is desirable.

Non-satiation conditions

Though non-satiation cannot be assumed to hold in general, it can under some circumstances. One such circumstance is as follows.

Proposition 10. ***Non-satiation in RCESE given no aggregate inferior goods***
Given an RCESE $(\bar{p}, \bar{\psi})$, suppose $\delta(\bar{p}, \bar{\psi}) \geq 0$. Then $\bar{\psi}^i \not\ll 0 \; \forall i$. If $v^i(\cdot)$ is locally nonsatiated for all i, then all i exhaust their budgets.

Proof. For all (p, ψ), we have $G(p, \psi)\delta(p, \psi) = 0$, and thus

$$\psi^i(p, \psi) \cdot \delta(p, \psi) = \nabla w^i(s(p)) \cdot Ds(p) G(p, \psi) \delta(p, \psi) = 0 \;\; \forall i. \tag{49}$$

Since $\delta(\bar{p}, \bar{\psi}) \geq 0$, it follows that $\psi^i(\bar{p}, \bar{\psi}) \not\ll 0$. For each i, because $v^i(\cdot)$ is locally nonsatiated, it follows that there is a good in which i's consumption utility is nonsatiated in equilibrium. Therefore all individuals exhaust their budgets. □

Proposition 10 tells us that as long as $\delta \geq 0$—that is, as long as each good ℓ is not "inferior in aggregate", in that x_ℓ is not locally decreasing in the profit rate—then each i can infer that it will be optimal for her to exhaust her budget. As a result, i does not need to choose her demands by constructing G, and then ψ^i, using known δ. Instead, i can recognize that *any* generalized inverse $\tilde{G}$ of $-Dz$ satisfies $-Dz(\bar{p})\tilde{G}dx^i = dx^i$ for $dx^i \perp \bar{p}$, and thus that $\tilde{G}$ captures the price impacts,[17] and pins down the supply impacts, of demand choices among baskets that exhaust i's budget. Therefore, in this setting,

$$x_{[\psi^i]}(p, \pi) = \operatorname*{argmax}_{x^i} \tilde{u}^i(x^i, \tilde{\psi}^i) \;\Big|\; p \cdot x^i = p \cdot e^i + \theta^i \pi \tag{50}$$

for any $\tilde{\psi}^i = \left(Ds(\bar{p})\tilde{G}\right)^T \nabla w^i(s(\bar{p}))$, where $\tilde{G}$ is an arbitrary generalized inverse of $-Dz(\bar{p})$.

Corollary 10.1. ***Non-satiation in RCESE given separability of consumption utility***
Suppose that, for all i, $v^i(\cdot)$ is additively separable and locally nonsatiated. Then, in any RCESE $(\bar{p}, \bar{\psi})$, $\bar{\psi}^i \not\ll 0 \; \forall i$, and all individuals exhaust their budgets.

Proof. The additive separability of the $v^i(\cdot)$ and thus of the $\tilde{u}^i(\cdot)$ implies that, at *any* (p, ψ), no good is inferior for any i. Therefore no good is inferior in aggregate. Recalling that $\delta(\bar{p}, \bar{\psi})$ must be defined if $(\bar{p}, \bar{\psi})$ is an RCESE, we have $\delta(\bar{p}, \bar{\psi}) \geq 0$. The result then follows from Proposition 10. □

Proposition 10 and its corollary thus demonstrate that, under some conditions on the economic primitives, an altruist does not need to know the gradient of the aggregate Engel curve in order to compute her optimal demands. It is possible that further inquiry will likewise identify reasonable conditions under which she can dispense with some of the price substitution variables. There thus appears to be at least some grounds for hope that, though theory alone offers no guidance to the altruistic consumer, in some circumstances the project of altruism in general equilibrium is feasible.

5 Conclusion

Spending decisions are sometimes motivated, in whole or part, by a desire to changing quantities supplied. In the simplest case, people engage in philanthropy. When

[17] Up to rescaling. Without loss of generality, we can restrict ourselves to price impacts with $dp_L = 0$ by imposing that the bottom row of $\tilde{G}$ equal $\mathbb{0}_L^T$.

individuals act on their preferences over supply—i.e., following Andreoni (1990), on their altruism—they engage in a public good game.

A conventional, non-altruistic consumer can choose her demands on the basis of her budget and the price vector, without consulting any further economic data. An altruist cannot. Standard models of public good games partially acknowledge this. When deciding how to allocate his budget across public goods, a philanthropist's best response to other philanthropists' spending decisions is recognized to depend not only on his own budget and the goods' prices but also on the aggregate quantities being provided by others. However, because prices and budgets are modeled as exogenous to the philanthropist's spending decisions, he is spared from having to consider the income and price substitution effects of other market participants. Like the conventional consumer, he can act as if the world is a vending machine that linearly transforms currency into goods. Unless the production possibility set is a simplex, so that relative prices are in fact exogenous, such analyses have been incomplete.

This paper and Kaufmann et al. (2024) have taken steps toward completing them by embedding altruists in a production economy. This has for the first time allowed a characterization of how optimal spending behavior by altruists depends on the broader array of relevant economic variables. In both papers, this characterization reveals that important behaviors that economic agents in a competitive setting are typically assumed to exhibit—in particular price-taking—cannot be assumed when the agents are altruistic.

The present paper further demonstrates that important behaviors exhibited in a simple partial equilibrium setting (or a conventional public good provision setting) cannot be assumed in a general equilibrium setting. One cannot even straightforwardly assume budget-exhaustion, nor a sign-preserving relationship between the preference for increasing the quantity of a good and the willingness to pay for a unit of it in the marketplace. A lesson for altruists is that to determine which purchases are optimal, or even desirable, one must gather and process much more economic data than has hitherto been assumed necessary or is typically done in practice: data almost analogous to what one would need to plan an economy. If one draws the pessimistic conclusion that predicting the impacts of small individual spending decisions on aggregate quantities is typically infeasible (and if one trusts that the effects of at least some kinds of policy are more often predictable), perhaps a broad further lesson is that resources spent on improving the world through philanthropic and consumer decisions are typically better directed to influencing policy.

Then by studying economies with multiple altruists, this paper and that of Kaufmann et al. independently identify the natural equilibrium concept that emerges from their interaction. In our setting, this lets us confirm that counterintuitive demand-adjustments can be optimal for all altruists at once.

If we maintain the hope that it is sometimes feasible to account for the general equilibrium effects of our spending decisions, a natural next step is to identify conditions under which the these effects can be approximated with less data than would be required to calculate the expressions given here. Such work would be valuable to individual altruistic consumers and philanthropists, and may make the associated behaviors more appealing by putting estimates of their consequences on surer footing. It may also shed light on the implications of philanthropy and ethical consumerism for social welfare and optimal policy in various settings beyond that considered by Kaufmann et al..

Complicating matters, on the other hand, further research is necessary to understand optimal behavior by individuals with preferences not directly over aggregate quantities of various goods (as most clearly in the case of concern for animal welfare) but over other individuals' consumption utility levels. Preferences of the latter kind can generate nonzero impacts associated with a given good purchase even in an exchange economy, since, as emphasized by Wilkinson (2022), purchases can affect the distribution of income.

Further research is also necessary to understand the impacts of marketplace decisions through channels other than that considered here.

For instance, though we have assumed a single, profit-maximizing firm, altruists may aim to direct capital away from firms whose production processes they see as doing harm (via divestment) or toward those doing good (via impact investing). These efforts obey a logic similar to the equilibrium displacement logic we have seen in the case of consumer spending.[18] It would evidently be straightforward, therefore, to introduce a model of optimal investment by agents with altruistic preferences along the lines of the model of optimal consumption introduced here. A unified model of the economic implications of altruism in both domains may also identify

---

[18]Because investors value diversification, the supply of capital each firm or industry receives is not perfectly elastic in the interest rate it offers. An individual investor's decision to increase or decrease his own investment in it therefore affects its size in equilibrium, even in an approximately competitive environment.

This point is made clearly by Christiano (2019), but to my knowledge, the financial economics literature on impact investing to date has neglected it. Instead either

a) investors are assumed to be infinitesimal and thus fully price-taking, and invest in ethical firms offering below-market returns only because they have an intrinsic preference for doing so (somewhat analogously to Dufwenberg et al. (2011), discussed in Section 1 of this paper: see e.g. Pastor et al. (2021) or Roth Tran (2019), beginning of Section IV);

b) an investor with strictly positive market power is considered (see e.g. Roth Tran (2019), end of Section IV, or Roth (2021)); or

c) firms are assumed to face a perfectly elastic supply of capital at a fixed interest rate, so that ethical investors can have an impact only by investing in projects offering below-market returns which otherwise would have received no investment and which therefore crowd out no investment (Green and Roth, 2024).

important interactions between them.

Moreover, we have studied a static economy, but the most significant consequences of spending decisions may arise only in a dynamic setting. For instance, increases in demand for a good today may result in better technology for producing the good in future periods, either directly through "learning by doing" or by motivating profit-seeking firms to research improved methods for producing goods they expect to remain in high demand. This suggests the value of embedding a model of altruism in a model of economic growth.

Finally, we have of course assumed a smooth, frictionless, and perfectly competitive environment. There are surely intuitive grounds for believing that frictions—e.g. the costs of altering production capacity for some good—in some sense do not alter the logic of continuous equilibrium displacement "on average" (see e.g. Isaacs et al. (2024)). Some (e.g. Budolfson (2019)) have disputed this, however, arguing informally that when there are adjustment costs, small consumers may have approximately no impact on supply even in expectation. This debate can only be adjudicated by carefully studying the frictions in question and their implications for the path between supply and marginal consumer behavior. If there is any lesson in the limited work done so far on the theory of strategic equilibrium displacement, it is that economic intuitions developed for other purposes can lead us astray. Effects of a purchase that might at first seem insignificant, such as infinitesimal price-changes or income effects induced in others, can be of primary importance to the altruistic consumer.

Acknowledgments

I thank attendees at the Oxford micro theory student seminar and the GPI work in progress group for feedback, and Kevin Roberts and Alex Teytelboym for extensive advice.

References

Andreoni, James, "Impure Altruism and Donations to Public Goods: A Theory of Warm-Glow Giving," *The Economic Journal*, 1990, *100* (1), 464–477.

Aumann, Robert J., "Markets with a Continuum of Traders," *Econometrica*, 1964, *32* (1–2), 39–50.

Bade, Sophie, "Nash Equilibrium in Games with Incomplete Preferences," *Economic Theory*, 2005, *26*, 309–332.

Bergstrom, Theodore, Lawrence Blume, and Hal Varian, "On the Private Provision of Public Goods," *Journal of Public Economics*, 1986, *29* (1), 25–49.

Budolfson, Mark Bryant, "The Inefficacy Objection to Consequentialism and the Problem with the Expected Consequences Response," *Philosophical Studies*, 2019, *176* (7), 1711–1724.

Christiano, Paul, "Analyzing divestment," https://sideways-view.com/2019/05/25/analyzing-divestment/ (accessed 2 Oct 2024), 2019. "The Sideways View", blog post.

Dufwenberg, Martin, Paul Heidhues, Georg Kirchsteiger, Frank Riedel, and Joel Sobel, "Other-Regarding Preferences in General Equilibrium," *Review of Economic Studies*, 2011, *78* (2), 613–39.

Ellickson, Bryan, *Competitive Equilibrium: Theory and Applications*, Cambridge, UK: Cambridge University Press, 1994.

Gardner, Bruce L., "Income Distribution through Multi-market Commodity Programs," in Bruce L. Gardner, ed., *The Economics of Agricultural Policies*, New York: Macmillan, 1987.

Green, Daniel and Benjamin Roth, "The Allocation of Socially Responsible Capital," *Journal of Finance*, 2024. Forthcoming.

Hosoya, Yuhki, "On the Uniqueness and Stability of the Equilibrium Price in Quasi-Linear Economies," 2022. Working paper.

Isaacs, Yoaav, Adam Lerner, and Jeffrey Sanford Russell, "Counting Your Chickens," *Australasian Journal of Philosophy*, 2024, *102* (3).

Jackson, Matthew and Alejandro Manelli, "Approximately Competitive Equilibria in Large Finite Economies," *Journal of Economic Theory*, 1997, *77* (2), 354–376.

Kaufmann, Marc, Peter Andre, and Botond Kőszegi, "Understanding Markets with Socially Responsible Consumers," *Quarterly Journal of Economics*, 2024, *139* (3), 1989–2035.

Konovalov, Alexander, "The Core of an Economy with Satiation," *Economic Theory*, 2005, *25*, 711–719.

Kreps, David M., *A Course in Microeconomic Theory*, Princeton, NJ: Princeton University Press, 1990.

_ , *Microeconomic Foundations I: Choice and Competitive Markets*, Princeton: Princeton University Press, 2012.

Mantel, Rolf, "On the Characterization of Aggregate Excess Demand," *Journal of Economic Theory*, 1974, *7* (3), 348–53.

Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green, *Microeconomic Theory*, Oxford: Oxford University Press, 1995.

Norwood, F. Bailey and Jayson L. Lusk, *Compassion, by the Pound: The Economics of Farm Animal Welfare*, Oxford: Oxford University Press, 2011.

Otani, Yoshihiko and Joseph Sicilian, "Limit Properties of Equilibrium Allocations of Walrasian Strategic Games," *Journal of Economic Theory*, 1990, *51* (2), 295–312.

Pastor, Lubos, Robert Stambaugh, and Lucian A. Taylor, "Sustainable Investing in Equilibrium," *Journal of Financial Economics*, 2021, *142* (2), 550–571.

Roberts, John and Andrew Postlewaite, "The Incentives for Price-taking Behavior in Large Exchange Economies," *Econometrica*, 1976, *44* (1), 115—-27.

Roth, Benjamin, "Impact Investing: A Theory of Financing Social Enterprises," 2021. Harvard Business School, Working Paper 20-078.

Roth Tran, Brigitte, "Divest, Disregard, or Double Down? Philanthropic Endowment Investments in Objectionable Firms," *American Economic Review: Insights*, 2019, *1* (2), 241–256.

Sobel, Joel, "Generous Actors, Selfish Actions: Markets with Other-Regarding Preferences," *International Review of Economics*, 2009, *56*, 3–16.

Varian, Hal R., *Microeconomic Analysis, Third Edition*, New York: W.W.Norton and Company, 1992.

Wilkinson, Hayden, "Market Harms and Market Benefits," *Philosophy & Public Affairs*, 2022. Online preprint.

Wohlgenant, Michael K., "Consumer Demand and Welfare in Equilibrium Displacement Models," in Jayson L. Lusk, Jutta Roosen, and Jason F. Shogren, eds., *The Oxford Handbook of the Economics of Food Consumption and Policy*, Oxford: Oxford University Press, 2011.

Appendices

A Supplemental materials

A.1 The normal form game

Players, strategies, and outcomes

The strategic interaction we study is fully identified by the economy $\mathcal{E}$. Up to one technical complication, detailed below, it may be defined as a normal form game as follows.

- Players: There are $I+1$ players, I individuals and nature.
- Strategy sets:
  - i's strategy set is $\mathbb{A}^i(\mathcal{E})$.
  - Nature chooses a continuous function $f(\cdot) : \mathbb{R}^{L-1} \to \mathbb{R}$.
- Outcomes: Denote the set of NRWES of $(\mathcal{E}, \{x^i(\cdot)\})$ by R. If

$$\hat{\hat{p}} \equiv \operatorname*{argmax}_{\hat{p} \in R} f(\hat{p})$$

  exists and is unique, $\hat{\hat{p}}$ the NRWES that obtains. All economic outcomes—supply, profits, individual demands, and thus individuals' utility levels—are determined by $\hat{\hat{p}}$, given $(\mathcal{E}, \{x^i(\cdot)\})$. Nature is indifferent among outcomes.

Because we have not defined the outcome of all strategy profiles, the game defined above is a normal form game with incomplete preferences. Following e.g. Bade (2005), a strategy profile of such a game is a Nash equilibrium if the corresponding utility profile $\{\bar{u}^i\}$ is defined and if, for each player i, no deviation yields a strategy profile at which u^i is defined and greater than $\bar{u}^i$.

In particular, we have not defined the outcome of a strategy profile in the event that there is no RWES of $(\mathcal{E}, \{x^i(\cdot)\})$. This is unnecessary because CESE, and the equilibrium refinement we consider in Section 3.2 (RCESE), presuppose strategy profiles compatible with an RWES.

We have also not defined the outcome of a strategy profile in the event that there are multiple NRWES and none uniquely maximizes $f(\cdot)$. The difficulty in this case is subtle, as detailed in the subsection just below, and is perhaps the reason that to my knowledge no existing literature on the strategic foundations of general equilibrium has presented the strategic interaction under study as a normal form game.

Selection of economic equilibrium

Suppose that $\hat{p}$ is an NRWES compatible with $(\mathcal{E}, \{x^i(\cdot)\})$ and that $\mathcal{E}$ is large relative to $\{x^i(\cdot)\}$. By Proposition 2, there is a unique NRWES near $\hat{p}$ compatible with the demand function profile $\{\tilde{x}^i(\cdot)\}$ in which a single i deviates from $\{x^i(\cdot)\}$.

Since a demand function profile may be compatible with multiple NRWES, we would like to say that nature identifies the *approximate* location of the NRWES, so that deviations by any individual produce only slight shifts to equilibrium prices. But there is no continuous, complete ordering on $\mathbb{R}^k$ for $k > 1$. If $L > 2$, therefore, there is no way to ensure that the NRWES nature chooses among those compatible with the demand function profile with i's deviation is always that NRWES near $\hat{p}$. Nature may be, as it were, on the fence between NRWES $\hat{p}$ and NRWES $\hat{p}'$, in which case a slight change to an individual's demand can discontinuously shift equilibrium prices.

To illustrate the difficulty, suppose that $L = 3$ and that nature orders NRWES lexicographically, choosing that with the highest value of p_1 if it is unique and, if multiple equally have the highest value of p_1, the NRWES among these with the highest value of p_2. The lexicographic ordering is strict; there are no ties. But a chosen NRWES is then not globally continuous in the demand function profile. If the NRWES compatible with $\{x^i(\cdot)\}$ are $\hat{p} = (1, 2)$ and $\hat{p}' = (1, 1)$, $\hat{p}$ is chosen; but if the NRWES compatible with $\{\tilde{x}^i(\cdot)\}$ are $\tilde{\hat{p}} = (1, 2)$ and $\tilde{\hat{p}}' = (1.01, 1)$, $\tilde{\hat{p}}'$ is chosen.

Under appropriate assumptions on $f(\cdot)$, however, we can ensure that, for any $m > 0$, almost all sets of m normalized price vectors have exactly one that maximizes $f(\cdot)$. For this reason, and to make our analysis as similar as possible to the existing literatures on equilibrium displacement and on the strategic foundations of general equilibrium, we ignore this complication in the body of the paper. Implicitly, we assume that whenever there is at least one NRWES of $(\mathcal{E}, \{x^i(\cdot)\})$, nature chooses one $\hat{p}$ among them; and that nature's strategy, fortuitously, is such that if any i deviates to a different demand function, the chosen NRWES is always that near to $\hat{p}$.

Relationship between CESE and Nash equilibrium

Call strategy profile $(\{x^i(\cdot)\}, f(\cdot))$ a CESE* of economy $\mathcal{E}$ if the implied NRWES $\hat{\hat{p}}$ is defined and $(\hat{\hat{p}}, \{x^i(\cdot)\})$ is a CESE.

CESE* is slightly more restrictive than CESE, in that it presupposes not only that there is at least one RWES of $(\mathcal{E}, \{x^i(\cdot)\})$ but also that a unique NRWES is identified by the economy and strategy profile $(\mathcal{E}, \{x^i(\cdot)\}, f(\cdot))$.

CESE* is a weakening of pure-strategy Nash equilibrium. Whereas Nash equilibrium maintains that no player can benefit by deviating, CESE* maintains only that the benefits to deviating a player would be able to achieve in an n-replicated economy and demand function profile fall to zero as $n \to \infty$.

A.2 Impacts in elasticity terms

For convenience in applications we can express (23) in elasticity terms, as long as $s(p) \gg 0$.

Given prices $p \gg 0$, let $\sigma(p)$ denote the matrix of cross-price elasticities of supply around prices p and $\varepsilon(p, x(\cdot))$ denote the matrix of cross-price elasticities of implicit demand around p given aggregate demand function $x(\cdot)$. Given an L-vector b, let $\mathrm{Diag}(b)$ denote the $L \times L$ diagonal matrix with the entries of b on its diagonal. Finally let

$$\phi(p, x(\cdot)) \equiv \mathrm{Diag}(p)^{-1}\, G(p, x(\cdot))\, \mathrm{Diag}(s(p)) \tag{A.1}$$

denote the generalized inverse of $(\sigma(p) - \varepsilon(p, x(\cdot)))$ with

$$\phi(\bar{p}, \bar{x}(\cdot))\, \mathrm{Diag}(s(p))^{-1}\, \delta(p, x(\cdot)) = 0$$

and whose bottom row consists of zeroes. Such a generalized inverse within the hyperplane perpendicular to p exists, is unique, and is equal to expression (A.1) because for $dx^i \perp p$ we have

$$\begin{aligned} & -Dz(p)\, G(p, x(\cdot))\, dx^i = dx^i \\ \Longrightarrow\ & \mathrm{Diag}(s(p))\, (\sigma(p) - \varepsilon(p, x(\cdot)))\, \mathrm{Diag}(p)^{-1}\, G(p, x(\cdot))\, dx^i = dx^i \\ \Longrightarrow\ & (\sigma(p) - \varepsilon(p, x(\cdot)))\, \phi(p, x(\cdot))\, \mathrm{Diag}(s(p))^{-1}\, dx^i = \mathrm{Diag}(s(p))^{-1}\, dx^i. \end{aligned}$$

(Observe that given $s(p) \gg 0$, $dx^i \perp p$ iff $\mathrm{Diag}(s(p))\, dx^i \perp p$.)

Substituting

$$\begin{aligned} G(p, x(\cdot)) &= \mathrm{Diag}(p)\, \phi(p, x(\cdot))\, \mathrm{Diag}(s(p))^{-1}, \\ Ds(p) &= \mathrm{Diag}(s(p))\, \sigma(p)\, \mathrm{Diag}(p)^{-1} \end{aligned}$$

into (23) yields

$$\begin{aligned} \psi^i(p, x(\cdot)) &= \big(\mathrm{Diag}(s(p))\, \sigma(p)\, \phi(p, x(\cdot))\, \mathrm{Diag}(s(p))^{-1}\big)^T \nabla w^i(s(p)) \\ &= \big(\sigma(p)\, \phi(p, x(\cdot))\big)^T \nabla w^i(s(p)). \end{aligned}$$

B Proofs

B.1 Proof of Proposition 1

Given economy $\mathcal{E}$ and aggregate demand function $x(\cdot) \in \mathbb{A}(\mathcal{E})$, consider $(\mathcal{E}^{(n)}, x^{(n)}(\cdot))$. Excess demand in the replicated setting equals

$$z^{(n)}(p) = nx(p, \pi^{(n)}(p, x^{(n)}(\cdot))/n) - ns(p), \tag{B.1}$$

where aggregate supply in the replicated economy equals $ns(p)$ because the aggregate endowment and production function have both been multiplied by n, and where

$$\begin{aligned}\pi^{(n)}(p, x^{(n)}(\cdot)) &\equiv \pi : x^{(n)}(p, \pi) = ns(p) \qquad \text{(B.2)}\\ &= \pi : x(p, \pi/n) = s(p)\\ &= n\big(\pi : x(p, \pi) = s(p)\big)\\ &= n\pi(p, x(\cdot)). \qquad \text{(B.3)}\end{aligned}$$

Substituting (B.3) into (B.1), we have $z^{(n)}(p) = nz(p)$. It follows immediately that p is a [R]WES of $(\mathcal{E}, x(\cdot))$ iff it is a [R]WES of $(\mathcal{E}^{(n)}, x^{(n)}(\cdot))$, for any $n \geq 1$.

Define

$$Z(p, \pi, x(\cdot)) \equiv p \cdot x(p, \pi) - p \cdot s(p).$$

Because $x(p, \pi)$ is h.o.d. 0 in (p, π) by admissibility and $s(p)$ is h.o.d. 0 in p, $Z(p, \pi, x(\cdot))$ is h.o.d. 0 in (p, π). In particular, if $Z(p, \pi, x(\cdot)) = 0$, $Z(kp, k\pi, x(\cdot)) = 0\ \forall k$. It follows that $\pi(p, x(\cdot))$ is is h.o.d. 1 in p.

Thus

$$\begin{aligned}z(kp) &= x(kp, \pi(kp, x(\cdot))) - s(kp)\\ &= x(kp, k\pi(p, x(\cdot))) - s(kp)\\ &= x(p, \pi(p, x(\cdot))) - s(p)\\ &= z(p);\end{aligned}$$

$z(p)$ is h.o.d. 0 in p. So any positive rescaling of $\bar{p}$ is also a WES. In particular, $\check{\hat{\bar{p}}}$ is a WES. Thus $\hat{\bar{p}}$ is an NWES.

Conversely, if $\hat{\bar{p}}$ is an NWES, any positive rescaling of $\check{\hat{\bar{p}}}$, including $\bar{p}$, is a WES.

B.2 Proof of Proposition 2

Let $\bar{p}$ be a regular WES of economy $\mathcal{E}$ and locally regular demand function profile $\{\bar{x}^i(\cdot)\} \in \mathbb{A}^i(\mathcal{E})$. Choose $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E})$ that is $\mathcal{C}^1$ around $(\bar{p}, \bar{\pi})$.

Let

$$h(\pi, \hat{p}, \alpha) \equiv \check{\hat{p}} \cdot \big(\bar{x}(\check{\hat{p}}, \pi) + \alpha\big(x^i(\check{\hat{p}}, \pi) - \bar{x}^i(\check{\hat{p}}, \pi)\big) - s(\check{\hat{p}})\big). \qquad \text{(B.4)}$$

Because $s(\cdot)$, $x^i(\cdot)$, and each $\bar{x}^i(\cdot)$ are $\mathcal{C}^1$ around $(\check{\hat{\bar{p}}}, \bar{\pi}/p_L)$, and because the composition of $\mathcal{C}^1$ functions is $\mathcal{C}^1$, $h(\cdot)$ is $\mathcal{C}^1$ around $(\bar{\pi}/p_L, \hat{\bar{p}}, \alpha)$ for all α.

By construction of the profit function, $h(\bar{\pi}, \hat{\bar{p}}, 0) = 0$. Also, by the local regularity of $\{\bar{x}^i(\cdot)\}$, $\partial h/\partial \pi$ is nonzero around $(\bar{\pi}, \hat{\bar{p}}, 0)$. Thus, by the IFT, there is a unique,

$\mathcal{C}^1$ function, which we will denote $\underline{\hat{\pi}}(\hat{p}, \alpha)$, such that $\underline{\hat{\pi}}(\hat{\bar{p}}, 0) = \pi(\check{\hat{\bar{p}}}, \bar{x}(\cdot)) \equiv \bar{\pi}$ and $h(\underline{\hat{\pi}}(\hat{p}, \alpha), \hat{p}, \alpha) = 0$ for all $(\hat{p}, \alpha)$ near $(\hat{\bar{p}}, 0)$.

Now, let

$$\underline{\hat{z}}(\hat{p}, \alpha) \equiv \mathcal{I}\big[\bar{x}(\check{\hat{p}}, \underline{\hat{\pi}}(\hat{p}, \alpha)) + \alpha\big(x^i(\check{\hat{p}}, \underline{\hat{\pi}}(\hat{p}, \alpha)) - \bar{x}^i(\check{\hat{p}}, \underline{\hat{\pi}}(\hat{p}, \alpha))\big) - s(\check{\hat{p}})\big]. \tag{B.5}$$

Because $s(\cdot)$, $x^i(\cdot)$, and each $\bar{x}^i(\cdot)$ are locally $\mathcal{C}^1$, and $\underline{\hat{\pi}}(\cdot)$ is $\mathcal{C}^1$, it follows that $\underline{\hat{z}}(\cdot)$ is locally $\mathcal{C}^1$ around $(\hat{\bar{p}}, 0)$.

Because $\bar{p}$ is a WES of $(\mathcal{E}, x(\cdot))$, $\underline{\hat{z}}(\hat{\bar{p}}, 0) = 0$. Also, because $\bar{p}$ is a regular WES, the Jacobian of $\underline{\hat{z}}(\cdot)$ with respect to prices is nonsingular at $(\hat{\bar{p}}, 0)$. Thus, by the IFT, there exist $\epsilon_1, \epsilon_2 > 0$ such that, for every $\alpha \in (-\epsilon_1, \epsilon_1)$, there is a unique $\hat{p} \in \mathcal{N}_{\epsilon_2}(\hat{\bar{p}})$ such that $\underline{\hat{z}}(\hat{p}, \alpha) = 0$.[19] Furthermore, defining

$$\hat{g}(\alpha) \equiv \hat{p} : \underline{\hat{z}}(\hat{p}, \alpha) = 0, \;\; \alpha \in (-\epsilon_1, \epsilon_1), \tag{B.6}$$

$\hat{g}(\cdot)$ is $\mathcal{C}^1$.

Thus

$$\underline{\hat{z}}(\hat{g}(1/n), 1/n) = 0 \;\; \forall n \geq \underline{n} \equiv \lfloor 1/\epsilon_1 \rfloor + 1. \tag{B.7}$$

It follows that $\hat{g}(1/n)$ is a normalized WES, and $(\hat{g}(1/n), 1)$ is a WES, of economy $\mathcal{E}$ and aggregate demand function

$$x(\cdot) \equiv \bar{x}(\cdot) + (x^i(\cdot) - \bar{x}^i(\cdot))/n. \tag{B.8}$$

It also follows that $\hat{g}(1/n)$ is the unique normalized WES in $\mathcal{N}_{\epsilon_2}(\hat{\bar{p}})$.

Finally, because $\underline{\hat{z}}(\cdot)$ is locally $\mathcal{C}^1$, and because the determinant of its Jacobian with respect to prices is nonzero at $(\hat{\bar{p}}, 0)$, there is an $\epsilon_3 > 0$ such that, for all $(\hat{p}, \alpha) \in \mathcal{N}_{\epsilon_3}((\hat{\bar{p}}, 0))$, the determinant of its Jacobian with respect to prices is nonzero at $(\hat{p}, \alpha)$. Therefore, as long as we pick ϵ_1 and ϵ_2 small enough that

$$\sqrt{\epsilon_1^2 + \epsilon_2^2} \leq \epsilon_3, \tag{B.9}$$

we can guarantee that, as long as $n \geq \lfloor 1/\epsilon_1 \rfloor + 1$, $\hat{g}(1/n)$ is not only a WES but a regular WES of $(\mathcal{E}, x(\cdot))$.

It then follows from Proposition 1 and the definition of n-replicated utility that, as long as $n \geq \lfloor 1/\epsilon_1 \rfloor + 1$, $\hat{g}(1/n)$ is a regular WES, and the unique normalized WES in $\mathcal{N}_{\epsilon_2}(\hat{\bar{p}})$, of $(\mathcal{E}^{(n)}, \bar{x}^{-i(n)}(\cdot) + x^{i(n)}(\cdot))$.

[19] $\mathcal{N}_\epsilon(b)$ denotes the open neighborhood of radius ϵ around vector b.

B.3 Proof of Proposition 3

Let $\bar{p}$ be a WES of economy $\mathcal{E}$ and locally regular demand function profile $\{\bar{x}^i(\cdot)\} \in \{\mathbb{A}^i(\mathcal{E})\}$.

Let n be large enough that $\mathcal{E}^{(n)}$ is large with respect to $\{\bar{x}^i(\cdot)\}^{(n)}$ and $\hat{\bar{p}}$. Then consider the change in $u_{\bar{p}}^{i(n)}$ that i achieves by deviating to locally $\mathcal{C}^1$ demand function $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E})$:

$$v^i\Big(x^{i(n)}\Big(\check{\hat{p}}_{\bar{p}}^{(n)}\big(x^i(\cdot)\big), \pi^{(n)}\big(\check{\hat{p}}_{\bar{p}}^{(n)}\big(x^i(\cdot)\big), \bar{x}^{-i(n)}(\cdot) + x^{i(n)}(\cdot)\big)\Big)\Big) + nw^i\Big(s\Big(\check{\hat{p}}_{\bar{p}}^{(n)}\big(x^i(\cdot)\big)\Big)\Big)$$
$$-v^i\big(\bar{x}^{i(n)}(\bar{p}, \pi^{(n)}(\bar{p}, \bar{x}^{(n)}(\cdot)))\big) - nw^i\big(s(\bar{p})\big). \tag{B.10}$$

In steps, we will take the limit of (B.10) as $n \to \infty$ and determine when the expression is nonpositive for any admissible choice of $x^i(\cdot)$.

First, by definitions (18) and (19), (B.10) equals

$$v^i\big(x^i\big(\check{\hat{p}}_{\bar{p}}^{(n)}\big(x^i(\cdot)\big), \pi\big(\check{\hat{p}}_{\bar{p}}^{(n)}\big(x^i(\cdot)\big), \bar{x}(\cdot) + (x^i(\cdot) - \bar{x}^i(\cdot))/n\big)\big)\big) - v^i\big(\bar{x}^i(\bar{p}, \bar{\pi})\big)$$
$$+nw^i\big(s\big(\check{\hat{p}}_{\bar{p}}^{(n)}\big(x^i(\cdot)\big)\big)\big) - nw^i\big(s(\bar{p})\big). \tag{B.11}$$

By Proposition 2 and the definition of $\hat{p}_{\bar{p}}^{(n)}(\cdot)$, we have

$$\check{\hat{p}}_{\bar{p}}^{(n)}(x^i(\cdot)) = g(1/n) \equiv (\hat{g}(1/n), 1), \tag{B.12}$$

where $\hat{g}(\cdot)$ is defined as in (B.6). Because $g(\alpha)$ is continuous and equals $\check{\hat{\bar{p}}}$ at $\alpha = 0$, and because $x^i(p, \pi)$ is h.o.d. 0 and $\pi(p, x(\cdot))$ is h.o.d. 1 in p, the limit as $n \to \infty$ of the first term of (B.11) equals

$$v^i(x^i(\bar{p}, \bar{\pi})).$$

Substituting (B.12) into the third term of (B.11), and replacing n with $1/\alpha$ (where $\alpha \equiv 1/n$), the third and fourth terms equal

$$\frac{w^i\big(s(g(\alpha))\big) - w^i\big(s(\bar{p})\big)}{\alpha}. \tag{B.13}$$

Because $w^i(\cdot)$, $s(\cdot)$, and $g(\cdot)$ are differentiable with $s(g(0)) = s(\bar{p})$, the limit of (B.13) as $\alpha \to 0$ equals

$$\frac{\partial}{\partial \alpha}\big[w^i(s(g(\alpha)))\big]_{\alpha=0},$$

which, by the chain rule, equals

$$\nabla w^i(s(\bar{p})) \cdot Ds(\bar{p}) \nabla g(0). \tag{B.14}$$

The first and second of these partial derivatives is given directly by the functions $w^i(\cdot)$ and $s(\cdot)$. We will now find the third: $\nabla g(0) = (\nabla \hat{g}(0), 0)$.

Recall the construction of $\hat{g}(\alpha)$ in (B.4)–(B.6). By the IFT,

$$\nabla \hat{g}(0) = -\Big(D\hat{\underline{z}}_{\hat{p}}(\hat{\bar{p}}, 0)\Big)^{-1} D\hat{\underline{z}}_{\alpha}(\hat{\bar{p}}, 0), \tag{B.15}$$

where $\hat{\underline{z}}(\hat{p}, \alpha)$ is defined as in (B.5) and $D\hat{\underline{z}}_{\hat{p}}(\hat{\bar{p}}, 0)$ and $D\hat{\underline{z}}_{\alpha}(\hat{\bar{p}}, 0)$ are the Jacobians of $\hat{\underline{z}}(\cdot)$ with respect to $\hat{p}$ and α, respectively, evaluated at $(\hat{\bar{p}}, 0)$.

Element ℓ, k of $D\hat{\underline{z}}_{\hat{p}}(\hat{\bar{p}}, 0)$ (defined for $\ell, k < L$) equals

$$\frac{\partial \bar{x}_\ell(\check{\hat{\bar{p}}}, \hat{\bar{\pi}})}{\partial p_k} + \frac{\partial \bar{x}_\ell(\check{\hat{\bar{p}}}, \hat{\bar{\pi}})}{\partial \pi} \frac{\partial \underline{\hat{\pi}}(\hat{\bar{p}}, 0)}{\partial \hat{p}_k} - \frac{\partial s_\ell(\check{\hat{\bar{p}}})}{\partial p_k}. \tag{B.16}$$

Element ℓ ($< L$) of $D\hat{\underline{z}}_{\alpha}(\hat{\bar{p}}, 0)$ equals

$$\frac{\partial \bar{x}_\ell(\check{\hat{\bar{p}}}, \hat{\bar{\pi}})}{\partial \pi} \frac{\partial \underline{\hat{\pi}}(\hat{\bar{p}}, 0)}{\partial \alpha} + dx^i_\ell, \tag{B.17}$$

where

$$dx^i \equiv x^i(\bar{p}, \bar{\pi}) - \bar{x}^i(\bar{p}, \bar{\pi}).$$

Likewise, recalling the construction of $\underline{\hat{\pi}}(\hat{p}, \alpha)$ following (B.4), and letting

$$\hat{\bar{\pi}} \equiv \pi(\check{\hat{\bar{p}}}, \bar{x}(\cdot)) \ \big(= \bar{\pi}/\bar{p}_L, \ = \underline{\hat{\pi}}(\hat{\bar{p}}, 0)\big),$$

the IFT gives us

$$\begin{aligned}
\frac{\partial \underline{\hat{\pi}}(\hat{\bar{p}}, 0)}{\partial \hat{p}_k} &= -\frac{1}{\bar{p} \cdot \hat{\bar{\delta}}} \sum_{\ell=1}^{L} \bar{p}_\ell \Big(\frac{\partial \bar{x}_\ell(\check{\hat{\bar{p}}}, \hat{\bar{\pi}})}{\partial p_k} - \frac{\partial s_\ell(\check{\hat{\bar{p}}})}{\partial p_k} \Big) \ \ (k < L), && \text{(B.18)} \\
\frac{\partial \underline{\hat{\pi}}(\hat{\bar{p}}, 0)}{\partial \alpha} &= -\frac{1}{\bar{p} \cdot \hat{\bar{\delta}}} \, \bar{p} \cdot dx^i, && \text{(B.19)}
\end{aligned}$$

where

$$\begin{aligned}
\bar{\delta} &\equiv \nabla_\pi \bar{x}(\bar{p}, \bar{\pi}), \\
\hat{\bar{\delta}} &\equiv \nabla_\pi \bar{x}(\check{\hat{\bar{p}}}, \hat{\bar{\pi}}) \ \ \big(= \bar{p}_L \bar{\delta} \big).
\end{aligned}$$

The denominator terms $\bar{p} \cdot \hat{\bar{\delta}}$ must be nonzero by the local regularity of $\bar{x}(\cdot)$.

Substituting (B.18) and (B.19) into (B.16) and (B.17), expression (B.15) can be rewritten

$$\nabla \hat{g}(0) = \Big[M\mathcal{I}\big(\hat{\bar{\delta}} \times \bar{p} - (\bar{p} \cdot \hat{\bar{\delta}}) I_L\big) \Big] dx^i,$$

and so

$$\nabla g(0) = \begin{bmatrix} M\mathcal{I}\big(\hat{\bar{\delta}} \times \bar{p} - (\bar{p} \cdot \hat{\bar{\delta}}) I_L\big) \\ \mathbb{0}_L^T \end{bmatrix} dx^i, \tag{B.20}$$

where

$$M \equiv \Big(\mathcal{I}\Big((\bar{p} \cdot \bar{\delta})\big(D\bar{x}_p(\check{\hat{\bar{p}}}, \hat{\bar{\pi}}) - Ds(\check{\hat{\bar{p}}})\big) - D(\bar{\delta})\big(D_p\bar{x}(\check{\hat{\bar{p}}}, \hat{\bar{\pi}}) - Ds(\check{\hat{\bar{p}}})\big)\bar{p} \times \mathbb{1}_L\Big)\mathcal{I}^T\Big)^{-1}.$$

Let G denote the matrix coefficient on dx^i in (B.20). Having established the linearity of $\nabla g(0)$ in dx^i, we will now provide an alternative characterization of G.

Define

$$\begin{aligned} \underline{\pi}(p, \alpha) &\equiv \pi\big(p, \bar{x}(\cdot) + \alpha(x^i(\cdot) - \bar{x}^i(\cdot))\big) \quad \big(= p_L\, \hat{\underline{\pi}}(\hat{p}, \alpha)\big), \\ \underline{\bar{x}}(p, \alpha) &\equiv \bar{x}\big(p, \underline{\pi}(p, \alpha)\big), \end{aligned}$$

and define $\underline{\bar{x}}^i(p, \alpha)$ and $\underline{x}^i(p, \alpha)$ likewise. Since $\hat{\underline{\pi}}(\hat{p}, \alpha)$, as defined following (B.4), is differentiable and $\pi(p, x(\cdot))$ is h.o.d. 1 in p, $\underline{\pi}(\cdot)$ is differentiable. Thus $\underline{\bar{x}}(p, \alpha)$, $\underline{\bar{x}}^i(p, \alpha)$, and $\underline{x}^i(p, \alpha)$ are differentiable.

Now observe that, for all $\alpha \leq 1/\underline{n}$, the following hold exactly:

$$\begin{aligned} \underline{\bar{x}}(g(\alpha), \alpha) + \alpha(\underline{x}^i(g(\alpha), \alpha) - \underline{\bar{x}}^i(g(\alpha), \alpha)) &= s(g(\alpha)), \\ \underline{\bar{x}}(\bar{p}, 0) &= s(\bar{p}) \\ \implies s(g(\alpha)) - s(\bar{p}) - \big(\underline{\bar{x}}(\bar{p}, 0) - \underline{\bar{x}}(g(\alpha), \alpha)\big) &= \alpha(\underline{x}^i(g(\alpha), \alpha) - \underline{\bar{x}}^i(g(\alpha), \alpha)). \end{aligned} \tag{B.21}$$

Dividing both sides of (B.21) by α and taking the limit as $\alpha \to 0$, we have

$$-Dz(\bar{p})\nabla g(0) + \bar{\delta}\Big(\frac{\partial \underline{\pi}(\bar{p}, 0)}{\partial \alpha}\Big) = dx^i. \tag{B.22}$$

Because $\nabla g(0) = G dx^i$ and $\partial \underline{\pi}(\bar{p}, 0)/\partial \alpha = \partial \hat{\underline{\pi}}(\hat{\bar{p}}, 0)/\partial \alpha$, we then have

$$-Dz(\bar{p})G dx^i - \frac{p \cdot dx^i}{p \cdot \bar{\delta}}\bar{\delta} = dx^i. \tag{B.23}$$

Thus, for $dx^i \perp \bar{p}$, we have

$$-Dz(\bar{p})G dx^i = dx^i. \tag{B.24}$$

Observe that $Ds(\bar{p})\bar{p} = 0$. If prices all rise in proportion to their current levels, then the price vector has simply been rescaled, and because $s(p)$ is h.o.d. 0, supply levels will not change. $D\underline{x}(\bar{p})\bar{p} = 0$ likewise, by the definition of $\underline{x}$ and the fact that $\chi(p)$ is h.o.d. 0. So it follows from the definition (10) of $z(p)$ that

$$Dz(\bar{p})\bar{p} = 0. \tag{B.25}$$

Also, for any dp *not* proportional to $\bar{p}$, we have $d\hat{p} = \hat{p} - \hat{\bar{p}} \neq \mathbb{0}_{L-1}$. Because $\bar{p}$ is a regular WES, $D\hat{z}(\hat{\bar{p}})$ is of full rank, so $D\hat{z}(\hat{\bar{p}})d\hat{p} \neq \mathbb{0}_{L-1}$. It follows that any marginal price-change dp not proportional to $\bar{p}$ induces a change to excess demands, and thus that $Dz(\bar{p})dp \neq \mathbb{0}_L$. Thus

$$\text{Rank}(Dz(\bar{p})) = L - 1. \tag{B.26}$$

Returning to (B.24), we can now conclude that G is a generalized inverse of $-Dz(\bar{p})$. Furthermore, from (B.23), at $dx^i = -\bar{\delta}$ we have

$$Dz(\bar{p})G\bar{\delta} + \bar{\delta} = \bar{\delta}. \tag{B.27}$$

So $G\bar{\delta}$ either equals $\mathbb{0}_L$ or is proportional to $\bar{p}$. But we know that the bottom row of G consists of zeroes, so the last entry of $G\bar{\delta}$ equals 0. So

$$G\bar{\delta} = \mathbb{0}_L. \tag{B.28}$$

We will now construct the unique generalized inverse G of $Dz(\bar{p})$ whose bottom row consists of zeroes and for which $G\bar{\delta} = \mathbb{0}_L$.

Let UNV^T denote a singular value decomposition of $Dz(\bar{p})$ with $N_{\ell\ell} \neq 0$ for $\ell < L$ and $N_{LL} = 0$. Because G is a generalized inverse of $Dz(\bar{p})$, we must by (B.26) have

$$G = V \begin{bmatrix} N_1^{-1} & A \\ B & C \end{bmatrix} U^T, \tag{B.29}$$

where N_1 is the $(L-1) \times (L-1)$ principal submatrix of N and A, B, and C are $(L-1) \times 1$, $1 \times (L-1)$, and 1×1 respectively.

Since V is invertible, (B.28) reduces to

$$\begin{bmatrix} N_1^{-1} & A \\ B & C \end{bmatrix} \bar{\bar{\delta}} = \mathbb{0}_L \tag{B.30}$$

$$\implies A_\ell = -\frac{\bar{\bar{\delta}}_\ell}{\bar{\bar{\delta}}_L} \frac{1}{N_{\ell\ell}}, \quad \ell < L, \text{ and} \tag{B.31}$$

$$C = -\frac{1}{\bar{\bar{\delta}}_L} \sum_{\ell=1}^{L-1} B_\ell \bar{\bar{\delta}}_\ell, \tag{B.32}$$

where $\bar{\bar{\delta}} \equiv U^T \bar{\delta}$.

We will now impose the constraint that the bottom row of G consists of zeroes. Since U^T is invertible, the bottom row of G consists of zeroes iff the bottom row of

$$V \begin{bmatrix} N_1^{-1} & A \\ B & C \end{bmatrix} \tag{B.33}$$

consists of zeroes. This in turn implies

$$B_\ell = -\frac{V_{L\ell}}{V_{LL}} \frac{1}{N_{\ell\ell}}, \quad \ell < L. \tag{B.34}$$

By (B.32), this also gives us C. G is thus fully constructed.

Substituting Gdx^i for $\nabla g(0)$ into (B.14), we have $\psi^i \cdot dx^i$, where

$$\psi^i = \big(Ds(\bar{p})G\big)^T \nabla w^i(s(\bar{p})). \tag{B.35}$$

Thus the limit of (B.10) as $n \to \infty$ equals

$$\big(v^i(x^i(\bar{p},\bar{\pi})) + \psi^i \cdot x^i(\bar{p},\bar{\pi})\big) - \big(v^i(\bar{x}^i(\bar{p},\bar{\pi})) + \psi^i \cdot \bar{x}^i(\bar{p},\bar{\pi})\big). \tag{B.36}$$

So $(\bar{p}, \{\bar{x}^i(\cdot)\})$ is a CESE iff $\bar{x}^i(\bar{p},\bar{\pi})$ maximizes $v^i(x^i(\bar{p},\bar{\pi})) + \psi^i \cdot x^i(\bar{p},\bar{\pi})$, among feasible $x^i(\bar{p},\bar{\pi})$, for all i.

B.4 Proof of Proposition 5

Continuous differentiability of demand given $\psi \not\ll 0$

Given $\psi^i \not\ll 0$ for all i, no individual is ever satiated. Thus $\pi(p,\psi) = p \cdot y(p)$ and may be denoted $\pi(p)$, and $\chi^i_{[\psi]}(\cdot)$ depends only on ψ^i and may be denoted $\chi^i_{[\psi^i]}(\cdot)$. We will now show that $\chi^i_{[\psi^i]}(p)$ is defined and locally $\mathcal{C}^1$ in p for all i, $\psi^i \not\ll 0$, $p \in P$.

By definition, $\chi^i_{[\psi^i]}(p)$ is the value of x^i that maximizes

$$\tilde{u}^i_{[\psi^i]}(x^i) = v^i(x^i) + \psi^i \cdot x^i \tag{B.37}$$

at the given p, subject to i's budget constraint. It follows from the strict concavity of $v^i(\cdot)$ that (B.37) is continuous and strictly quasiconcave in x^i. Since i's budget set is compact for each $p \gg 0$, and $p \gg 0$ for all $p \in P$, $\chi^i_{[\psi^i]}(p)$ exists and is unique.

Letting $\tilde{x}^i_{[\psi^i]}(\cdot)$ denote i's demand as a function of prices and wealth B^i, $\tilde{x}^i_{[\psi^i]}(p, B^i)$ is $\mathcal{C}^1$ with respect to both arguments around (p, B^i) if

$$H \equiv \begin{bmatrix} D^2\tilde{u}^i_{[\psi^i]}\big(\tilde{x}^i_{[\psi^i]}(p,B^i)\big) & p \\ p^T & 0 \end{bmatrix} \tag{B.38}$$

is nonsingular (Kreps (2012), Proposition 11.10). ($\psi^i \not\ll 0$ implies the required local non-satiation.) Given an $(L+1)$-vector d, the bottom row of H and $p \gg 0$ ensure that $[Hd]_{L+1} = 0$ only if the first L entries of d are all zero or contain both positives and negatives. By additive separability and since

$$\frac{\partial^2 \tilde{u}^i_{[\psi^i]}}{\partial (x^i_\ell)^2}(x^i) = v^{i\prime\prime}_\ell(x^i_\ell) < 0 \quad \forall x^i, \tag{B.39}$$

$[Hd]_\ell = 0$ for $\ell \leq L$ only if d_ℓ and d_{L+1} are of the same sign. So $[Hd] = 0$ iff $d = 0$.

Since

$$B^i = p \cdot e^i + \theta^i \pi(p),$$

$x^i_{[\psi^i]}(p, \pi)$ is $\mathcal{C}^1$ in p. It then follows from $\chi^i_{[\psi^i]}(p) \equiv x^i_{[\psi^i]}(p, \pi(p))$, the definition of $\pi(p)$, and the continuous differentiability of supply across P that $\chi^i_{[\psi^i]}(p)$ is $\mathcal{C}^1$ in p under the stated conditions.

Existence of WES given $\psi \not\ll 0$

Our assumptions guarantee that, unless $\psi^i \ll 0 \; \forall i$,

a) $s(\cdot)$ is uniformly smooth and quasi-monotonic on a compact, convex set of price vectors $P \subset \Delta^{L-1}$;

b) $v^i(\cdot)$ is additively separable, strictly increasing, differentiable, and weakly concave for all i;

c) $x_{[\psi]}(p, \pi) \gg 0 \; \forall p \gg 0$ (because for each good ℓ, $\exists i : \lim_{x^i_\ell \to 0} v^{i\prime}(x^i_\ell) = \infty, \; e^i \neq 0$);

d) $\exists i : \psi^i \not\ll 0, \; \theta^i > 0$; and

e) $x_{[\psi]}(\cdot)$ is defined, admissible, and continuous.

We will now show that there is a WES of $(\mathcal{E}, x_{[\psi]}(\cdot))$ and that all WES p satisfy $\tilde{p} \in P^\circ$. (The proof roughly follows the pattern of the proof of Mas-Colell et al. (1995), Proposition 17.C.1.)

Under the conditions above, for simplicity fix $x(\cdot) = x_{[\psi]}(\cdot)$, and drop the ψ subscripts on other terms likewise. By (d), $x(\cdot)$ is nonsatiated, so $\pi(p, x(\cdot))$ is defined for all p.

Consider the excess demand function

$$z(p) = x(p, \pi(p, x(\cdot))) - s(p).$$

We will first show that $z(p)$ is continuous in p throughout P. This will follow directly from showing that $\pi(p, x(\cdot))$ is continuous in p.

Recall from (6) that

$$\pi(p, x(\cdot)) = \pi : \ Z(p, \pi, x(\cdot)) = 0,$$

where

$$Z(p, \pi, x(\cdot)) \equiv p \cdot x(p, \pi) - p \cdot s(p).$$

Suppose by contradiction that $\pi(p, x(\cdot))$ is not continuous in p. Then there exists a $\bar{p}$ and a $\gamma > 0$ such that, for all $\epsilon > 0$,

$$\exists p \in \mathcal{N}_\epsilon(\bar{p}) \ : \ \pi(p, x(\cdot)) \notin \big(\pi(\bar{p}, x(\cdot)) - \gamma, \pi(\bar{p}, x(\cdot)) + \gamma\big).$$

For every natural $n > 0$, we can choose a $p^n \in \mathcal{N}_{1/n}(\bar{p})$ such that either

$$\pi(p^n, x(\cdot)) \geq \pi(\bar{p}, x(\cdot)) + \gamma \text{ or}$$
$$\pi(p^n, x(\cdot)) \leq \pi(\bar{p}, x(\cdot)) - \gamma.$$

There must thus be either an infinite subset ν of the naturals such that

$$\pi(p^n, x(\cdot)) \geq \pi(\bar{p}, x(\cdot)) + \gamma \ \ \forall n \in \nu \tag{B.40}$$

or one such that

$$\pi(p^n, x(\cdot)) \leq \pi(\bar{p}, x(\cdot)) - \gamma \ \ \forall n \in \nu. \tag{B.41}$$

Let ν be an infinite subset of the naturals such that (B.40) holds, and consider the sequence $\{(p^n, \pi(\bar{p}, x(\cdot)) + \gamma)\}_{n \in \nu}$. Because $Z(p, \pi, x(\cdot))$ is strictly increasing in π, and $\pi : Z(p^n, \pi, x(\cdot)) = 0$ is no less than $\pi(\bar{p}, x(\cdot)) + \gamma$, we have

$$Z\big(p^n, \pi(\bar{p}, x(\cdot)) + \gamma, x(\cdot)\big) \leq 0 \ \ \forall n \in \nu. \tag{B.42}$$

Because $Z(p, \pi, x(\cdot))$ is continuous in p, however, and because $p^n \to \bar{p}$,

$$\{Z\big(p^n, \pi(\bar{p}, x(\cdot)) + \gamma, x(\cdot)\big)\}_{n \in \nu} \to Z(\bar{p}, \pi(\bar{p}, x(\cdot)) + \gamma) > 0, \tag{B.43}$$

with the inequality holding because $Z(\bar{p}, \pi(\bar{p}, x(\cdot)) + \gamma) = 0$ and $Z(p, \pi, x(\cdot))$ strictly increases in π. (B.42) contradicts (B.43), so there is no infinite subset of the naturals such that (B.40) holds.

Analogous reasoning proves that there is no infinite subset of the naturals such that (B.41) holds.

Therefore $\pi(p, x(\cdot))$ is continuous in p. So $z(p)$, as the composition of functions continuous throughout P, is also continuous in p throughout P.

We will now construct a correspondence $f(\cdot)$ from P to P, show that it has a fixed point $\bar{p}$, and then show that $\bar{p}$ is a WES.

For $p \in P^\circ$, let

$$f(p) = \{q \in P \ : \ \tilde{z}(p) \cdot q \geq \tilde{z}(p) \cdot q' \ \ \forall q' \in P\},$$

where

$$\tilde{z}(p) \equiv (I_L s(p))^{-1} z(p).$$

For $p \in P^c$, $\{1, ..., L\}$ can be partitioned into two nonempty sets of goods: $L_0(p) \equiv \{\ell : s_\ell(p) = 0\}$ and its complement $L_1(p)$. Let

$$Q(p) \equiv \cup_{d \in \Delta^{L-1}_{L_0(p)}} \operatorname*{argmax}_{q \in P} q \cdot d.$$

That is, $Q(p)$ is the set of $q \in P$ that maximize a weighted average of the prices of the goods in $L_0(p)$. By the compactness of P, $Q(p) \neq \emptyset$. By the quasi-monotonicity of $s(\cdot)$, $p \notin Q(p)$.

We will now show that $Q(p)$ is closed. Given a sequence $q^n \to q^*$ with $q^n \in Q(p)$ $\forall n$, since $\Delta^{L-1}_{L_0(p)}$ is compact, there is a convergent sequence $d^n \to d^*$ with $d^n \in \Delta^{L-1}_{L_0(p)}$ $\forall n$, $d^* \in \Delta^{L-1}_{L_0(p)}$, and

$$q^n \in \operatorname*{argmax}_{q \in P} q \cdot d^n \ \ \forall n. \tag{B.44}$$

If $q^* \notin \operatorname{argmax}_{q \in P} q \cdot d^*$, then, since P is compact, there is a $q' \in P, \notin \Delta^{L-1}_{L_0(p)}$ with $q' \cdot d^* > q^* \cdot d^*$. Then by the continuity of $q \cdot d$ in q and d, we have $q' \cdot d^n > q^n \cdot d^n$ for n sufficiently large, contradicting (B.44).

Choose a homeomorphism $\phi(\cdot)$ between P^c and the unit $(L-1)$-sphere. Let

$$\epsilon(p) \equiv \min_{q \in Q(p)} ||\phi(q) - \phi(p)|| > 0,$$
$$h(p) \equiv \{p' \in P^c \ : \ ||\phi(p') - \phi(p)|| < \epsilon(p)\}.$$

Finally, let

$$f(p) = P^c \backslash h(p).$$

For all $p \in P$, $f(p)$ is a topological disc. If $p \in P^\circ$, this follows from the fact that $f(p)$ is a level set of a linear function defined on the compact, convex set P, and so is compact and convex. If $p \in P^c$, this follows from the facts that the boundary of a closed disc (as P is) is a topological sphere, that $h(p)$ is a hole in P^c (since the image $\phi(h(p))$ is a hole in the unit $(L-1)$-sphere), and that a topological sphere with a hole removed is a topological disc.

To show that $f(\cdot)$ is upper hemicontinuous, consider a sequence $(p^n, q^n) \to (p, q)$, all within P, with $q^n \in f(p^n)$ $\forall n$.

If $p \in P^\circ$, then $p^n \in P^\circ$ for n sufficiently large. It follows from $q^n \cdot \tilde{z}(p^n) \geq q' \cdot \tilde{z}(p^n)\ \forall q' \in P$ and the continuity of $\tilde{z}(\cdot)$ across P° that $q \in f(p)$.

If $p \in P^c$, then there is either a subsequence whose p-elements are contained in P°, or one whose p-elements are contained in P^c. Let such a subsequence be indexed by m.

In the first case, by the continuity of $z(\cdot)$ and $s(\cdot)$ on P, and the fact that $\chi(p) \gg 0\ \forall p \in P$, we have $\lim_{m\to\infty} \tilde{z}_\ell(p^m) = \infty$ if $\ell \in L_0(p)$, with the limit finite otherwise. If $q \notin Q(p)$, there is a $q' \in P$ with $q'_\ell \geq q_\ell$ for all $\ell \in L_0(p)$ and $q'_\ell > q_\ell$ for some $\ell \in L_0(p)$. Thus for m sufficiently large, $\tilde{z}(p^m) \cdot q' > \tilde{z}(p^m) \cdot q^m$, contradicting the stipulation that $q^m \in f(p^m)$. So $q \in Q(p) \subset f(p)$.

In the second case, we can choose a subsequence $\{(p^m, q^m)\}$ such that, for m sufficiently large, $L_0(p^m)$ is constant and, by the continuity of $s(\cdot)$ on P, a weak subset of $L_0(p)$. Then for m sufficiently large, $q^m \in Q(p^m) \subseteq Q(p) \subseteq f(p)$. Since $Q(p)$ is closed, $q \in Q(p) \subset f(p)$.

We have established that $f(\cdot)$ is a disc-valued, upper hemicontinuous correspondence from a nonempty, compact, convex set P to itself. By Kakutani's fixed point theorem, there is a $\bar{p} \in P$ with $\bar{p} \in f(\bar{p})$.

We cannot have $\bar{p} \in P^c$, since $s(\bar{p}) \not\gg 0$ but $\chi(\bar{p}) \gg 0$ for $\bar{p} \in P^c$. Given $\bar{p} \in P^\circ$, we must have $z(\bar{p}) = 0$. Otherwise there are ℓ, k with $z_\ell(\bar{p}) > 0$, $z_k(\bar{p}) < 0$. But $z(\bar{p}) \cdot \bar{p} = 0$, by construction of the profit function; and since $\bar{p} \in P^\circ$ there is an $\epsilon > 0$ such that $p \equiv \bar{p} + \epsilon(a_\ell - a_k) \in P$ and $\tilde{z}(\bar{p}) \cdot p > 0$, where in general a_ℓ denotes the unit vector with a one in place ℓ. This contradicts $\bar{p} \in f(\bar{p})$. So $\bar{p}$ is a WES.

Also, more generally, observe that all WES must be in P°, to ensure positive supply.

Regularity of all WES given $\psi \not\ll 0$

Let $\lambda^i_{[\psi^i]}(p)$ denote the marginal utility of nominal wealth for i at p given ψ^i. Since for each ℓ

$$\lambda^i_{[\psi^i]}(p) = \frac{1}{p_\ell}\Big(v^{i\prime}_\ell(\chi^i_{[\psi^i]\ell}(p)) + \psi^i_\ell\Big), \tag{B.45}$$

the continuous differentiability of $\chi^i_{[\psi^i]}(\cdot)$ and twice continuous differentiability of $v^i_\ell(\cdot)$ imply that $\lambda^i_{[\psi^i]}(\cdot)$ is $\mathcal{C}^1$ in p under the stated conditions as well.

Let

$$\tilde{\Psi} \equiv \big\{\psi : \psi^i \not\ll 0\ \forall i,\ \psi^i = \psi^j\ \forall i, j\big\}$$

denote the set of ψ with ψ^i identical and not entirely negative for all i. We will now show that $Dz_{[\psi]}(p)dp \neq 0$ for all $p \in P$, $dp \not\propto p$, and $\psi \in \tilde{\Psi}$. This will establish that $D\hat{z}_{[\psi]}(\hat{p})$ is invertible for all (p, ψ) with $\tilde{p} \in P$ and $\psi \in \tilde{\Psi}$.

The lower Inada condition rules out corner solutions, so $\chi^i_{[\psi^i]}(p)$ satisfies, for the given p and ψ^i,

$$Dv^i\big(\chi^i_{[\psi^i]}(p)\big)^T + \psi^i = \lambda^i_{[\psi^i]}(p)p. \tag{B.46}$$

Differentiating (B.46) with respect to p yields

$$\begin{aligned} D^2v^i\big(\chi^i_{[\psi^i]}(p)\big)D\chi^i_{[\psi^i]}(p) &= pD\lambda^i_{[\psi^i]}(p) + \lambda^i_{[\psi^i]}(p)I_L \\ \implies D\chi^i_{[\psi^i]}(p) &= D^2v^i\big(\chi^i_{[\psi^i]}(p)\big)^{-1}\Big(pD\lambda^i_{[\psi^i]}(p) + \lambda^i_{[\psi^i]}(p)I_L\Big). \end{aligned} \tag{B.47}$$

Fixing p, choose a price-change vector $dp \not\propto p$, and let

$$\tilde{dp}_{[\psi^i]} \equiv dp - \frac{D\lambda^i_{[\psi^i]}(p)\cdot dp}{D\lambda^i_{[\psi^i]}(p)\cdot p}p.$$

Note that the denominator represents the change in the marginal utility of wealth for i per unit change of prices in the direction of p: that is, per unit of positive rescaling of the price vector. A positive rescaling of the price vector leaves χ^i unchanged, proportionally increasing both i's budget and all prices, proportionally lowering the marginal utility of nominal wealth (which positive by $\psi^i \gg 0$). So the denominator is negative, and in particular nonzero.

Since $D\chi^i_{[\psi^i]}(p)p = 0$, $D\chi^i_{[\psi^i]}(p)\tilde{dp}_{[\psi^i]} = D\chi^i_{[\psi^i]}(p)dp$. Also, by construction, $\tilde{dp}_{[\psi^i]}$ is an eigenvector of the second factor on the right-hand side of (B.47), with

$$\Big(pD\lambda^i_{[\psi^i]}(p) + \lambda^i_{[\psi^i]}(p)I_L\Big)\tilde{dp}_{[\psi^i]} = \lambda^i_{[\psi^i]}(p)\tilde{dp}_{[\psi^i]}.$$

Then, since the corresponding eigenvalue is $\lambda^i_{[\psi^i]}(p) > 0$, and since by additive separability and strict concavity $D^2v^i\big(\chi^i_{[\psi^i]}(p)\big)^{-1}$ is a diagonal matrix with negative entries, (B.47) gives us that $D\chi^i_{[\psi^i]}(p)dp$ is a vector whose entries have signs opposite to those of $\tilde{dp}_{[\psi^i]}$.

Since the normalized Jacobian of supply $D\hat{s}(\hat{p})$ is positive semidefinite, and since by the nonsingularity assumption $Ds(p)\tilde{dp}_{[\psi^i]} \neq 0$ for $\tilde{dp}_{[\psi^i]} \not\propto p$, $Ds(p)\tilde{dp}_{[\psi^i]}$ preserves the sign of at least one nonzero entry of $\tilde{dp}_{[\psi^i]}$.

Given symmetry of preferences and budgets, $\psi \in \tilde{\Psi}$ implies that $D\chi_{[\psi]}(p)$ is proportional to $D\chi^i_{[\psi^i]}(p)$ for each i. This completes the proof that $Dz_{[\psi]}(p)dp \neq 0$ under the stated conditions.

Bounding ψ

Let $\tilde{dp}_{[\psi]}$ denote $\tilde{dp}_{[\psi^i]}$ for any i (given $\psi \in \tilde{\Psi}$ and thus ψ^i the same for all i), and let

$$\mathcal{P} \equiv \{dp : ||dp|| = 1, dp_L = 0\}$$

denote the set of price-change vectors that leave the price of good L fixed and whose absolute value is 1. We will now show that

$$\underline{m} \equiv \inf_{p \in P, dp \in \mathcal{P}, \psi \in \tilde{\Psi}} \left[\max_{\ell} \left| [-Dz_{[\psi]}(p)dp]_{\ell} \right| \right] > 0.$$

If not, there is a sequence $p^{(n)}, dp^{(n)}, \psi^{(n)}$ in $P \times \mathcal{P} \times \tilde{\Psi}$ such that

$$\lim_{n \to \infty} -Dz_{[\psi^{(n)}]}(p^{(n)})dp^{(n)} = 0. \tag{B.48}$$

Because P and $\mathcal{P}$ are compact, there is a subsequence of this sequence for which $p^{(n)}$ and $dp^{(n)}$ converge to some p^*, dp^* and limit (B.48) is maintained. Also, for each n in this subsequence, there is a unique $\kappa^{(n)} \in \mathbb{R}$ such that

$$\tilde{dp}^{(n)}_{[\psi^{(n)}]} = dp^{(n)} + \kappa^{(n)} p^{(n)}.$$

We can therefore choose a sequence—a subsequence of this subsequence of the original sequence—for which $p^{(n)}$ and $dp^{(n)}$ converge, limit (B.48) is maintained, and either

a) $\kappa^{(n)} \to \kappa^* \in (-\infty, \infty)$ or

b) $\kappa^{(n)} \to \infty$ or $\kappa^{(n)} \to -\infty$.

Choose such a sequence. Because $Ds(p)dp$ is continuous in both arguments, (B.48) implies

$$\lim_{n \to \infty} D\chi_{[\psi^{(n)}]}(p^{(n)})dp^{(n)} = \lim_{n \to \infty} Ds(p^{(n)})dp^{(n)} = Ds(p^*)dp^*.$$

So for each nonzero element of $Ds(p^*)dp^*$ (of which there is at least one, since $dp^* \not\propto p^*$), the sign of the corresponding element of $D\chi_{[\psi^{(n)}]}(p^{(n)})dp^{(n)}$ is the same for all sufficiently large n. But for each n, there is an ℓ such that $[Ds(p^{(n)})dp^{(n)}]_\ell$ is nonzero and of the same sign as $[\tilde{dp}^{(n)}_{[\psi^{(n)}]}]_\ell$, and so (equivalently) of opposite sign to $[D\chi_{[\psi^{(n)}]}(p^{(n)})dp^{(n)}]_\ell$. Let $\ell^{(n)} > 0$ denote the absolute value of the element(s) among these that is (are) largest in absolute value. (B.48) implies that $\ell^{(n)} \to 0$. We will now show that this is incompatible with both cases (a) and (b).

In case (a), $\ell^{(n)} \to 0$ implies

$$\lim_{n \to \infty} \tilde{dp}^{(n)}_{[\psi^{(n)}]} \cdot Ds(p^{(n)}) \tilde{dp}^{(n)}_{[\psi^{(n)}]} = (dp^* + \kappa^* p^*) Ds(p^*)(dp^* + \kappa^* p^*) \leq 0.$$

This implies that $D\hat{s}(\hat{p})$ is not positive definite. This is impossible, since as the Jacobian of a supply function it is positive semidefinite and symmetric, and by assumption it is nonsingular.

In case (b), suppose $\kappa^{(n)} \to \infty$. Then there is an $\underline{n}$ such that $\tilde{dp}^{(n)}_{[\psi^{(n)}]} \gg 0 \; \forall n \geq \underline{n}$. So $D\chi_{[\psi^{(n)}]}(p^{(n)})dp^{(n)} \ll 0 \; \forall n \geq \underline{n}$. So each entry of $Ds(p^*)dp^*$ is weakly negative, with at least one entry strictly negative. This is impossible, since $p^* \cdot Ds(p^*)dp^* = 0$ (recalling that $p^* \cdot Ds(p^*) = 0$, by symmetry of $Ds(p) \; \forall p$). The $\kappa^{(n)} \to -\infty$ case is ruled out analogously.

Let a_ℓ denote the unit vector with a one in place ℓ. Suppose $dx^i = a_\ell$, and observe that $||dx^i|| = 1$. For any $p \in P$ and $\psi \in \tilde{\Psi}$, there is a unique dp with $dp_L = 0$ satisfying

$$-Dz_{[\psi]}(p)dp = dx^i$$

(where $G_{[\psi]}(p)$ is the generalized inverse of $-Dz_{[\psi]}(p)$ defined by (22), and $G_{[\psi]}(p)dx^i = dp + \kappa p$ for some κ). We must have $||dp|| \leq 1/\underline{m}$; if greater, then there is an entry of $-Dz_{[\psi]}(p)dp$ with absolute value greater than $\underline{m}(1/\underline{m}) = 1$, so

$$|| - Dz_{[\psi]}(p)dp|| = ||dx^i|| > 1,$$

a contradiction. Since

$$\begin{aligned} \psi^i_\ell(p, \psi)^T &= Dw^i(s(p))Ds(p)G(p, \psi)a_\ell \\ &= Dw^i(s(p))Ds(p)dp, \end{aligned} \tag{B.49}$$

where the second expression is continuous in p and dp, and since P and $\{dp : dp_L = 0, ||dp|| \leq 1/\underline{m}\}$ are compact, there is a bound $\bar{\bar{\psi}} \gg 0$ such that

$$\psi^i(p, \psi) \in [-\bar{\bar{\psi}}, \bar{\bar{\psi}}]^L \;\; \forall p \in P, \, \psi \in \tilde{\Psi}.$$

Furthermore, the additive separability of all utility functions guarantees that $\delta(p, \psi) \geq 0$, and $\psi \in \tilde{\Psi}$ implies condition (d) listed under "Existence of WES", which guarantees that $\delta(p, \psi) \neq 0$. There is thus for each i some $dx^i = \delta(p, \psi) > 0$ such that $G(p, \psi)x^i = 0$, and therefore such that $\psi^i(p, \psi) \cdot dx^i = 0$. We cannot have $\psi^i(p, \psi) \ll 0$ for any i. So, letting Ψ denote the subset of $\tilde{\Psi}$ with $\psi^i \in [-\bar{\bar{\psi}}, \bar{\bar{\psi}}]^L \; \forall i$, we have

$$\psi(p, \psi) \in \Psi \;\; \forall p \in P, \, \psi \in \Psi. \tag{B.50}$$

Observe that Ψ is compact and a topological disc.

Existence of RCESE

At any price vector p, all consumers have the same budgets and quasi-utility functions, and thus buy the same baskets. Under this constraint, the strict quasiconcavity of $\tilde{u}^i_{\psi^i}(\cdot)$ for any ψ and the strict convexity of the production set imply that a unique aggregate supply vector s^* is efficient. By the first fundamental theorem of welfare economics, if $\bar{p}$ is a Walrasian equilibrium, $s(\bar{p}) = s^*$.

Suppose there are price vectors $p \in P^\circ$, $p' \neq p, \in P^\circ$ with $s(p) = s(p') = s^*$. Then for all $\alpha \in [0, 1]$,

$$\operatorname*{argmax}_s \, ((1-\alpha)p + \alpha p') \cdot s = s^*,$$

since s^* maximizes both $p \cdot s$ and $p' \cdot s$. Thus $Ds(p)(p' - p) = 0$, implying that $D\hat{s}(\hat{p})$ is singular at p. But since p and p' are both in P, we cannot have $p' - p \propto p$. By definition of uniform smoothness, it follows that there is a unique $p \in P^\circ$ with $s(p) = s^*$.

So there is a unique WES $\bar{p} \in P^\circ$ for each impact matrix $\psi \in \Psi$. Denote it by $p(\psi)$. We will now show that $p(\cdot)$ is $\mathcal{C}^1$ across Ψ.

First we will show that $\chi_{[\psi]}(p)$ is $\mathcal{C}^1$ in both arguments throughout $P \times \Psi$. Recall that we have already established that $\chi_{[\psi]}(p)$ is $\mathcal{C}^1$ in p for any $\psi \in \Psi$. Also, recall that $\psi \in \Psi$ guarantees budget-exhaustion for all individuals, so that we can write $\chi^i_{[\psi^i]}(p)$, noting that $\chi^i_{[\cdot]}(\cdot)$ is continuous in prices, because $x^i_{[\psi^i]}(p, \pi)$ is continuous in (p, π) and $\pi = p \cdot y(p)$ is continuous in p.

Suppose that, for some i, $\chi^i_{[\psi^i]}(p)$ is not even continuous in (p, ψ) throughout $P \times \Psi$. Then there is a sequence $(p^{(n)}, \psi^{i(n)}) \to (p, \psi^i)$, all within $P \times \Psi$, such that $\chi^i_{[\psi^{i(n)}]}(p^{(n)})$ converges to some χ^{i*} (by the Bolzano-Weierstrass theorem and the fact that maximal demands for i are bounded across all budgets supported by $p \in P$) and

$$\chi^i_{[\psi^i]}(p) \neq \chi^{i*}. \tag{B.51}$$

Since utility-maximizing demands are single-valued, $\tilde{u}^i_{[\cdot]}(\cdot)$ is continuous in both arguments, and $\chi^i_{[\cdot]}(\cdot)$ is continuous in prices, (B.51) implies that, for n sufficiently large,

$$\tilde{u}^i_{[\psi^{i(n)}]}\big(\chi_{[\psi^i]}(p^{(n)})\big) > \tilde{u}^i_{[\psi^{i(n)}]}\big(\chi_{[\psi^{i(n)}]}(p^{(n)})\big),$$

contradicting the requirement that $\chi_{[\psi^{i(n)}]}(p^{(n)})$ be maximize $\tilde{u}^i_{[\psi^{i(n)}]}$ at prices $p^{(n)}$. So $\chi^i_{[\cdot]}(\cdot)$ is continuous in both arguments for all i, and the sum $\chi_{[\cdot]}(\cdot)$ is as well.

Given a marginal shift $d\psi^i$ to i's impact vector, holding prices and profits fixed, i's demands x^i maintain the two optimality conditions

1. $\frac{1}{p_\ell}\frac{\partial \tilde{u}^i}{\partial x^i_\ell}$ equal for all ℓ,
2. $p \cdot x^i = B^i$

if they shift marginally by dx^i satisfying

$$\begin{aligned} D^2v^i(x^i)dx^i + d\psi^i &= kp \\ \implies dx^i &= \left(D^2v^i(x^i)\right)^{-1}(kp - d\psi^i) \quad \text{for some } k, \text{ and} \qquad \text{(B.52)} \\ p \cdot dx^i &= 0. \qquad \text{(B.53)} \end{aligned}$$

Combining (B.52) and (B.53) yields

$$\begin{aligned} p \cdot \left(D^2v^i(x^i)\right)^{-1}(kp - d\psi^i) &= 0 \\ \implies k &= \frac{p \cdot \left(D^2v^i(x^i)\right)^{-1}d\psi^i}{p \cdot \left(D^2v^i(x^i)\right)^{-1}p}. \qquad \text{(B.54)} \end{aligned}$$

Substituting (B.54) into (B.52) and simplifying yields

$$dx^i = \left(1 - \frac{1}{p \cdot \left(D^2v^i(x^i)\right)^{-1}p}\right)\left(D^2v^i(x^i)\right)^{-1}d\psi^i. \qquad \text{(B.55)}$$

Since (B.55) is defined for all $p \gg 0$ (so for all $p \in P$) and x^i with $x^i_{[\psi^i]}(p,\pi)$ (for some $\psi \in \Psi$ and $\pi \geq 0$), $x^i_{[\psi^i]}(p,\pi)$ is differentiable in ψ^i on $P \times \Psi$, with the Jacobian given by the coefficient on $d\psi^i$ on the right-hand side of (B.55). Because $x^i_{[\psi^i]}(p,\pi) = \chi^i_{[\psi^i]}(p)$ is continuous in p and ψ^i, and $v^i(\cdot)$ is $\mathcal{C}^2$ in x^i, this Jacobian is continuous in (p,π). This establishes that the partial derivatives of $\chi^i_{[\psi^i]}(p)$ with respect to ψ^i exist and are continuous in (p,ψ^i).

The last step is to establish that the partial derivatives of $\chi^i_{[\psi^i]}(p)$ with respect to p are also continuous in (p,ψ^i) (and not just p in isolation). This follows from immediately from (B.47) and the continuous differentiability of $\lambda^i_{[\psi^i]}(p)$ in both arguments, which in turn follows from (B.45) and the continuity of $\chi^i_{[\psi]}(p)$ in both arguments.

So $\chi_{[\psi]}(p)$ is $\mathcal{C}^1$ in both arguments across $P \times \Psi$.

Because $s(\cdot)$ is $\mathcal{C}^1$ across P and $\chi_{[\cdot]}(\cdot)$ is $\mathcal{C}^1$ in both arguments across $P \times \Psi$, $z_{[\cdot]}(\cdot)$ is $\mathcal{C}^1$ in both arguments across $P \times \Psi$. So therefore is $\hat{z}_{[\cdot]}(\cdot)$, where, following (14),

$$\hat{z}_{[\psi]}(\hat{p}) \equiv \mathcal{I}z_{[\psi]}(\check{\hat{p}}). \qquad \text{(B.56)}$$

Recall also that $D_{\hat{p}}\hat{z}_{[\psi]}(\hat{p})$ is invertible for any $(\hat{p},\psi)$ for which $\hat{p}$ is a regular normalized WES given ψ. By the IFT, this establishes that if $p(\psi)$ is defined, $p(\cdot)$ is defined and continuous (in fact $\mathcal{C}^1$) throughout a neighborhood of ψ in Ψ. The continuity of $z_{[\cdot]}(\cdot)$ in both arguments then implies that $p(\psi)$ is defined and continuous for all $\psi \in \Psi$.

By (B.50) and the fact that $\tilde{p} \in P$ for all WES p,

$$\psi^*(\psi) \equiv \psi(p(\psi), \psi)$$

is a function from the nonempty compact topological disc Ψ to itself. We will now show that $\psi^*(\cdot)$ is continuous.

We will first show that $G(p(\psi), \psi)$ is continuous in ψ throughout Ψ.

Suppose it is not. Then there exist $\psi, \psi' \in \Psi$ such that

$$\lim_{\epsilon \to 0} G\big(p(\psi^{(\epsilon)}), \psi^{(\epsilon)}\big) \neq G\big(p(\psi), \psi\big),$$

where

$$\psi^{(\epsilon)} \equiv \psi + \epsilon(\psi' - \psi).$$

There must then be some $dx^i \in \mathbb{R}^L$ such that

$$\lim_{\epsilon \to 0} G\big(p(\psi^{(\epsilon)}), \psi^{(\epsilon)}\big) dx^i \neq G\big(p(\psi), \psi\big) dx^i. \tag{B.57}$$

Denote the left- and right-hand sides by g and g^* respectively. By construction, $g_L = g^*_L = 0$. There is therefore no $\kappa \neq 0$ such $g = g^* + \kappa p(\psi)$. Then since $z_{[\cdot]}(\cdot)$ is locally $\mathcal{C}^1$ in both arguments around $(p(\psi), \psi)$, and since $p(\psi)$ is an RWES compatible with impact matrix ψ, (B.57) in turn implies

$$\lim_{\epsilon \to 0} -Dz_{[\psi^{(\epsilon)}]}\big(p(\psi^{(\epsilon)})\big) G\big(p(\psi^{(\epsilon)}), \psi^{(\epsilon)}\big) dx^i \neq \lim_{\epsilon \to 0} -Dz_{[\psi^{(\epsilon)}]}\big(p(\psi^{(\epsilon)})\big) G\big(p(\psi), \psi\big) dx^i.$$

This is impossible: the term in the limit on the left-hand side equals dx^i for all ϵ, by definition of G, and the right-hand limit equals dx^i because $p(\cdot)$ is continuous and $-z_{[\cdot]}(\cdot)$ is locally $\mathcal{C}^1$ in both arguments around $(p(\psi), \psi)$.

By the definition of ψ, the continuous differentiability of $w^i(s)$ in s for all i, and the continuous differentiability of $s(p)$ in p throughout P, $\psi(\psi)$ is continuous. The function $\psi(\psi)$ is thus a continuous function from the nonempty compact topological disc Ψ to itself. By Brouwer's fixed point theorem, it has a fixed point. By construction, for any such fixed point $\bar{\psi}$, $(p(\bar{\psi}), \bar{\psi})$ is an RCESE.

B.5 Proof of Proposition 6

The proof largely mirrors, and refers to, that in Appendix B.4. Let

$$v^{i*}_\ell \equiv \lim_{x^i_\ell \to \infty} v^{i\prime}_\ell(x^i_\ell),$$

$$\underline{v}^i_L(\psi^i) \equiv \max\left(0, \max_\ell \left(\psi^i_\ell + v^{i*}_\ell\right)\right),$$

$$\tilde{\Psi} \equiv \{\psi \,:\, \psi^i_L = 0 \;\forall i, \;\; w^i(\cdot) = 0 \implies \psi^i = 0 \;\forall i\}.$$

Continuous differentiability of demand given $\psi \in \tilde{\Psi}$ and high $\{v_L^i\}, \{B^i\}$

If $\psi \in \tilde{\Psi}$ and $v_L^i > 0$, no individual is ever satiated. Thus $\pi(p, \psi) = p \cdot y(p)$ and may be denoted $\pi(p)$, and $\chi^i_{[\psi]}(\cdot)$ depends only on ψ^i and may be denoted $\chi^i_{[\psi^i]}(\cdot)$.

We will now show that for all (p, ψ) with $p \in P$ and $\psi \in \tilde{\Psi}$, for each i, if $v_L^i > \underline{v}_L^i(\psi^i)$, there is a budget $\underline{B}^i(p, \psi^i, v_L^i)$ such that if $B^i > \underline{B}^i(p, \psi^i, v_L^i)$, then $\chi^i_{[\psi^i]}(p)$ is defined and locally $\mathcal{C}^1$ in p.

Recall that $\chi^i_{[\psi^i]}(p)$ is the value of x^i that maximizes $\tilde{u}^i_{[\psi^i]}$ (see (B.37)) at the given p, subject to i's budget constraint. It follows from the quasilinearity of $v^i(\cdot)$ in L, the strict concavity of $v^i_\ell(\cdot)$ $\forall \ell < L$, and $\psi^i_L = 0$ that (B.37) is continuous and strictly quasiconcave in x^i and thus that $\chi^i_{[\psi^i]}(p)$ is defined.

Define $\tilde{x}^i_{[\psi^i]}(\cdot)$ as in Appendix B.4. The lower Inada condition on $v^i_\ell(\cdot)$ for $\ell < L$ ensures that $\tilde{x}^i_{\ell[\psi^i]}(\cdot)(p, B^i) > 0$ for all $p \gg 0$, $B^i > 0$, $\ell < L$. The $v_L^i > \underline{v}_L^i(\psi)$ condition ensures that for each i and each $p^* \gg 0$ there is a budget

$$\underline{B}^i(p, \psi^i, v_L^i)$$

such that if $B^{i*} > \underline{B}^i(p, \psi^i, v_L^i)$, $\tilde{x}^i_{L[\psi^i]}(p, B^i) > 0$ for all (p, B^i) near (p^*, B^{i*}). The $v_L^i > 0$ and $\psi_L^i = 0$ conditions imply local non-satiation.

Then $\tilde{x}^i_{[\psi^i]}(p, B^i)$ is $\mathcal{C}^1$ with respect to both arguments around (p, B^i) if (B.38) is nonsingular. Given an $(L+1)$-vector d, the bottom row of H and $p \gg 0$ ensure that $[Hd]_{L+1} = 0$ only if the first L entries of d are all zero or contain both positives and negatives. By additive separability and since (B.39) holds for all $\ell < L$, $[Hd]_\ell = 0$ for $\ell < L$ only if d_ℓ and d_{L+1} are of the same sign. Also, since

$$\frac{\partial^2 \tilde{u}^i_{[\psi^i]}}{\partial (x_L^i)^2}(x^i) = 0 \quad \forall x^i,$$

$[Hd]_L = 0$ iff $d_L = 0$. So $[Hd] = 0$ iff $d = 0$.

It then follows as in Appendix B.4 that $\chi^i_{[\psi^i]}(p)$ is locally $\mathcal{C}^1$ in p if $\psi \in \tilde{\Psi}$, $v_L^i > \underline{v}_L^i(\psi^i)$ $\forall i$, and $B^i > \underline{B}^i(p, \psi^i, v_L^i)$ $\forall i$.

Finally, fixing v_L^i, let us show that $\underline{B}^i(\cdot)$ is continuous in its first two arguments, under the constraint that $\underline{v}_L^i(\psi) < v_L^i$. $\underline{B}^i(p, \psi, v_L^i)$ is the minimum value of B^i that sets

$$v_1^{i\prime}(\tilde{x}^i_{1[\psi^i]}(p, B^i)) + \psi_1^i = v_L^i, \tag{B.58}$$

and so the unique value of B^i that would maintain (B.58) if good L could not be purchased. Dropping good L from consideration, $\tilde{x}^i_{[\psi^i]}(\cdot)$ is $\mathcal{C}^1$ in prices and wealth, by the same proof of continuous differentiability as in Appendix B.4 (now with $L-1$ goods). A proof precisely analogous to that establishing that $\chi^i_{[\cdot]}(\cdot)$ is $\mathcal{C}^1$ in both arguments (see (B.51)–(B.55) and the surrounding discussion) then establishes that $\tilde{x}^i_{[\cdot]}(\cdot)$ is $\mathcal{C}^1$ in all three arguments (again, under the restriction that only the first $L-1$ goods may be purchased). By the IFT, $\underline{B}^i(\cdot)$ is continuous in p and ψ^i.

By continuity in p, we can define

$$\underline{b}^i(\psi, v^i_L) \equiv \min_{p \in P} \underline{B}^i(p, \psi, v^i_L).$$

Existence of WES given $\psi \in \tilde{\Psi}$ and high $\{v^i_L\}, \{B^i\}$

Our assumptions guarantee that, if $\psi^i_L = 0\ \forall i$, $v^i_L > \underline{v}^i_L(\psi)$, and $B^i > \underline{b}^i(\psi, v^i_L)$, conditions (a)–(e) under the "Existence of WES given $\psi \not\ll 0$" heading of Appendix B.4 are satisfied.

The proof that $z_{[\psi]}(\cdot)$ is defined and continuous throughout P is as in Appendix B.4. If $s(\cdot)$ is quasi-monotonic on P, the proof that a WES exists is also as in Appendix B.4. Suppose therefore that for all ℓ,

$$\exists i : w^i(\cdot) = 0, e^i_\ell > 0 \tag{B.59}$$

(and do not assume that $s(\cdot)$ is quasi-monotonic on P).

Because the production possibility set (and thus the supply possibility set) is compact,

$$\exists \bar{s} > 0 : s_\ell(p) < \bar{s} \ \ \forall \ell \ \ \forall p \gg 0. \tag{B.60}$$

It then follows from non-negativity of demands that

$$z_\ell(p) > -\bar{s} \ \ \forall \ell \ \ \forall p \gg 0. \tag{B.61}$$

Consider a sequence of positive prices $\{p^n\} \to p \neq 0$ such that $p_k = 0$ for some k. Choose an ℓ with $p_\ell > 0$, and choose an individual i such that condition (B.59) is satisfied for ℓ, so that

$$\lim_{n\to\infty} p^n \cdot e^i + \theta^{i\ell}\pi(p^n) > 0.$$

It follows from the additive separability and strict monotonicity of $v^i(\cdot)$ that

$$\lim_{n\to\infty} \chi^i_{k[\psi]}(p^n) = \infty.$$

Since demands by other individuals cannot be negative, and supply levels are bounded above, we have

$$\lim_{n\to\infty} \left\{ \max_k \left(z_{k[\psi]}(p^n) \right) \right\} = \infty. \tag{B.62}$$

It follows by Mas-Colell et al. (1995), Proposition 17.C.1 that $\exists \bar{p} \gg 0 : z(\bar{p}) = 0$. By definition, $\bar{p}$ is a WES.

Since demands are positive, $\tilde{\bar{p}} \in P^\circ$.

Regularity of all WES given $\psi \in \tilde{\Psi}$ and high $\{v^i_L\}, \{B^i\}$

If $\psi^i_L = 0\ \forall i$, $v^i_L > \underline{v}^i_L(\psi)$, and $B^i > \underline{b}^i(\psi, v^i_L)$, then $p \in P$ implies $\chi^i_{L[\psi^i]}(p) > 0\ \forall i$. Thus the marginal utility of nominal wealth for i is

$$\frac{v^{i\prime}_\ell(\chi^i_{[\psi]}(p))}{p_\ell} = \frac{v^i_L}{p_L} \quad \forall \ell. \tag{B.63}$$

For $\ell < L$, differentiating with respect to p_ℓ yields

$$\frac{\partial \chi^i_{\ell[\psi^i]}(p)}{\partial p_\ell} = \frac{v^i_L}{p_L\, v^{i\prime\prime}_\ell(\chi^i_{\ell[\psi^i]}(p))} < 0. \tag{B.64}$$

For $k < L$, $k \neq \ell$, differentiating with respect to p_k yields

$$\frac{\partial \chi^i_{\ell[\psi^i]}(p)}{\partial p_\ell} = 0. \tag{B.65}$$

The upper-left $(L-1) \times (L-1)$ submatrix of $D\chi^i_{[\psi^i]}(p)$ thus has negative entries on the diagonal and zeroes elsewhere.

For completeness, to verify that $D\chi^i_{[\psi^i]}(p)$ is defined, we can identify its bottom row and right column as follows. For $\ell < L$, differentiating the budget exhaustion condition

$$p \cdot \chi^i_{[\psi^i]}(p) = p \cdot e^i + \theta^i(p \cdot y(p))$$

with respect to p_ℓ yields

$$\sum_{k=1}^{L} p_k \frac{\partial \chi^i_{k[\psi^i]}(p)}{\partial p_\ell} = e^i_\ell + \theta^i y_\ell(p). \tag{B.66}$$

Substituting (B.64) and (B.65) into (B.66) and rearranging yields, for $\ell < L$,

$$\frac{\partial \chi^i_{L[\psi^i]}(p)}{\partial p_\ell} = -\frac{v^i_L p_\ell}{p^2_L\, v^{i\prime\prime}_\ell(\chi^i_{\ell[\psi^i]}(p))} + \frac{1}{p_L}(e^i_\ell + \theta^i y_\ell(p)).$$

Differentiating (B.63) with respect to L yields

$$\frac{\partial \chi^i_{\ell[\psi^i]}(p)}{\partial p_L} = -\frac{1}{v^{i\prime\prime}_\ell(\chi^i_{\ell[\psi^i]}(p))} \frac{v^i_L p_\ell}{p^2_L}$$

which in combination with (B.66) yields

$$\frac{\partial \chi^i_{L[\psi^i]}(p)}{\partial p_L} = \frac{v^i_L}{p^3_L} \sum_{\ell=1}^{L-1} \frac{p^2_\ell}{v^{i\prime\prime}_\ell(\chi^i_{\ell[\psi^i]}(p))} + \frac{1}{p_L}(e^i_L + \theta^i y_L(p)). \tag{B.67}$$

Fixing p, choose a price-change vector $dp \not\propto p$, and let

$$\tilde{dp} \equiv dp - dp_L\, p/p_L \tag{B.68}$$

Observe that $\tilde{dp}_L = 0$. $D\chi^i_{[\psi^i]}(p)dp$ is thus a vector whose entries have signs opposite to those of $\tilde{dp}$. Since this holds for all i, $D\chi^i_{[\psi]}(p)dp$ also has signs opposite to those of $\tilde{dp}$.

As in Appendix B.4, $Ds(p)dp = Ds(p)\tilde{dp}$ preserves the sign of at least one nonzero entry of $\tilde{dp}$. This completes the proof that $Dz_{[\psi]}(p)dp \neq 0$ under the stated conditions.

Bounding ψ

There is a $\bar{\bar{\psi}} \gg 0$ such that, for all $(p, \psi) \in P \times \tilde{\Psi}$, if $v^i_L > \underline{v}^i_L(\psi)$ and $B^i > \underline{b}^i(\psi, v^i_L)$,

$$\psi^i(p, \psi) \in [-\bar{\bar{\psi}}, \bar{\bar{\psi}}]^L.$$

The proof proceeds precisely as in Appendix B.4. We need only let $\tilde{dp}$ notationally take the place of $\tilde{dp}_{[\psi]}$, since in this quasilinear setting the analogous adjustment to dp (defined by (B.68)) is independent of ψ.

Let Ψ denote the compact subset of $\tilde{\Psi}$ with $\psi^i \in [-\bar{\bar{\psi}}, \bar{\bar{\psi}}]\ \forall i$. Since $\underline{v}^i_L(\cdot)$ is continuous, we can define

$$\underline{\underline{v}}^i_L \equiv \min_{\psi \in \Psi}\ \underline{v}^i_L(\psi).$$

Since $\underline{B}^i(\cdot)$ is continuous in p and ψ^i, we can define

$$\underline{\underline{B}}^i \equiv \max_{p \in P, \psi \in \Psi} \underline{B}^i(p, \psi^i, \underline{\underline{v}}^i_L).$$

Since $\underline{B}^i(\cdot)$ decreases in v^i_L, $B^i \geq \underline{\underline{B}}^i$ implies $B^i > \underline{B}^i(p, \psi^i, v^i_L) \geq \underline{b}^i(\psi, v^i_L)$ for any $p \in P$, $\psi \in \Psi$, $v^i_L > \underline{\underline{v}}^i_L$.

Here, $B^i \geq \underline{\underline{B}}^i$ and $v^i_L > \underline{\underline{v}}^i_L$ for all i, $p \in P$, and $\psi \in \Psi$ guarantee that all marginal wealth is spent on good L: $\delta(p, \psi) = a_L/p_L > 0$. So $G(p, \psi)a_L = 0$: purchases of good L do not affect prices and so do not affect supply. (Instead they only lower profits, lowering others' purchases of good L.) So $\psi^i_L(p, \psi) = 0\ \forall i$.

So, if $B^i \geq \underline{\underline{B}}^i$ and $v^i_L > \underline{\underline{v}}^i_L$ for all i,

$$\psi(p, \psi) \in \Psi \ \ \forall p \in P,\ \psi \in \Psi. \tag{B.69}$$

Observe that Ψ is a compact topological disc.

Existence of RCESE

Assume henceforth that $B^i \geq \underline{\underline{B}}^i$ and $v^i_L > \underline{\underline{v}}^i_L$ for all i.

In this setting, there is a WES of $(\mathcal{E}, x_{[\bar{\psi}]}(\cdot))$. It is unique up to rescaling. This follows from the fact that, for each i, $x_{[\bar{\psi}]}(\cdot)$ maximizes a quasilinear (quasi-)utility function: see Hosoya (2022), Theorem 1. Hosoya assumes that utility in goods $\ell < L$ is nondecreasing, whereas we allow marginal (quasi-)utility in such goods to be negative at sufficiently large values of x^i_ℓ, because we may have $\psi^i_\ell < -v^{i*}_\ell$. However, because each i's demands are identical to those that would obtain if marginal utility in each ℓ equaled 0 (rather than a negative number) at such large values of x^i_ℓ, Hosoya's result is maintained.

Let $p(\psi)$ denote the unique WES in P compatible with impact matrix $\psi \in \Psi$.

We will now show that $\chi_{[\psi]}(p)$ is $\mathcal{C}^1$ in both arguments throughout $P \times \Psi$. Recall that we have already established that $\chi_{[\psi]}(p)$ is $\mathcal{C}^1$ in p for any $\psi \in \Psi$. Continuity can be proven as it is surrounding (B.51) in the setting of Appendix B.4.

Given a marginal shift $d\psi^i$ to i's impact vector with $d\psi^i_L = 0$, holding prices and profits fixed, i's demands maintain the two optimality conditions

1. $\frac{1}{p_\ell}\frac{\partial \tilde{u}^i}{\partial x^i_\ell} = \frac{v^i_L}{p_L}$ for all ℓ,

2. $p \cdot x^i = B^i$

if they shift marginally by dx^i satisfying

$$
\begin{aligned}
D^2 v^i(x^i) dx^i + d\psi^i &= 0, \\
p \cdot dx^i &= 0 \\
\implies \; dx^i_\ell &= \frac{1}{v^{i\prime\prime}_\ell(\chi^i_{\ell[\psi^i]}(p))} d\psi^i_\ell, \quad \ell < L; \\
dx^i_L &= \frac{1}{p_L}\sum_{\ell=1}^{L-1} \frac{p_\ell}{v^{i\prime\prime}_\ell(\chi^i_{\ell[\psi^i]}(p))} d\psi^i_\ell.
\end{aligned}
$$

Thus $\chi^i_{[\psi^i]}(p)$ is differentiable in ψ^i. Since $\chi^i_{\ell[\psi^i]}(p)$ is continuous in both arguments for all $\ell < L$, and $v^i_\ell(\cdot)$ is $\mathcal{C}^2$ $\forall \ell < L$, the derivative is continuous in (p, ψ^i).

The last step is to establish that the partial derivatives of $\chi^i_{[\psi^i]}(p)$ with respect to p are also continuous in (p, ψ^i) (and not just p in isolation). This follows from immediately from (B.64)–(B.67) and the continuity of $\chi^i_{[\psi]}(p)$ in both arguments.

The proof that $\psi^*(\psi) \equiv \psi(p(\psi), \psi)$ has a fixed point $\bar{\psi}$ such that $(p(\bar{\psi}), \bar{\psi})$ is an RCESE concludes as in Appendix B.4.

B.6 Proof of Proposition 9

Choose a profile of I $\mathcal{C}^1$ functions $\{w^i(\cdot)\}$ from $\mathbb{R}^L_{++}$ to $\mathbb{R}$, with $I \geq 2L+1$ and $L \geq 2$; L-vectors $\bar{s} \gg 0$, $\bar{p} \gg 0$; and an $L \times L$ matrix M with $p \in \text{Null}(M)$ and $\bar{p} \notin \text{Col}(M)$. Let

$$R \equiv \text{Rank}(M).$$

Define ψ by $\psi^i = M \nabla w^i(\bar{s})\ \forall i$.

Choose a positive semidefinite, symmetric, $L \times L$ matrix N with $\text{Col}(N) = \text{Col}(M^T)$ (and thus $\text{Rank}(N) = R$), $N\bar{p} = \mathbb{0}_L$, and $\bar{p} \cdot N = \mathbb{0}_L^T$.

There is a (not necessarily unique) matrix G such that $NG = M^T$, $\text{Rank}(G) = L-1$, and $G_{L\ell} = 0\ \forall \ell$, i.e. the bottom row of G consists of zeroes. To construct one, let $m_\ell \equiv M^T a_\ell$ denote the ℓ^{th} column of M^T. Since $m_\ell \in \text{Col}(M^T)$, $m_\ell \in \text{Col}(N)$. It follows that, for each ℓ, there is a vector $\tilde{g}_\ell$ with

$$N\tilde{g}_\ell = m_\ell.$$

Choose $\{\tilde{g}_\ell\}_{\ell=1}^L$. Each can be decomposed into the sum of an element of $\text{Null}(N)$ and a vector $\tilde{\tilde{g}}_\ell$ orthogonal to $\text{Null}(N)$. So $\{\tilde{\tilde{g}}_\ell\}_{\ell=1}^L$ is a set of vectors orthogonal to $\text{Null}(N)$ such that

$$N\tilde{\tilde{g}}_\ell = m_\ell\ \ \forall \ell.$$

R of these vectors are linearly independent; if fewer, then $\text{Rank}(G) < R$ and thus $\text{Rank}(M) < R$. Choose R linearly independent elements of $\{\tilde{\tilde{g}}_\ell\}$, and for these, define $\tilde{\tilde{\tilde{g}}}_\ell = \tilde{\tilde{g}}_\ell$. For each of the $L-R$ elements remaining, define $\tilde{\tilde{\tilde{g}}}_\ell$ by adding a distinct element of a basis of the $(L-R)$-dimensional space $\text{Null}(N)$ to $\tilde{\tilde{g}}_\ell$. We now have a set of linearly independent vectors $\tilde{\tilde{\tilde{g}}}_\ell$ with

$$N\tilde{\tilde{\tilde{g}}}_\ell = m_\ell\ \ \forall \ell.$$

Construct G by setting its ℓ^{th} column equal to $g_\ell \equiv \tilde{\tilde{\tilde{g}}}_\ell - [\tilde{\tilde{\tilde{g}}}_\ell]_L / \bar{p}_L\, \bar{p}$. By construction, $G_{L\ell} = 0\ \forall \ell$. Since $\bar{p} \in \text{Null}(N)$, $Ng_\ell = m_\ell\ \forall \ell$; $NG = M^T$.

Since $Ga_L = 0$, $\text{Rank}(G) < L$. To verify that $\text{Rank}(G) \geq L-1$, observe that

$$G = \tilde{\tilde{\tilde{G}}} - \bar{p}\gamma^T,$$

where $\tilde{\tilde{\tilde{G}}}$ is the $L \times L$ matrix with column ℓ equaling $\{\tilde{\tilde{\tilde{g}}}_\ell\}$ and γ is the L-dimensional column vector with $\gamma_\ell = [\tilde{\tilde{g}}_\ell]_L / \bar{p}_L$. Because $\tilde{\tilde{\tilde{G}}}$ is of full rank, there is a vector $\tilde{\gamma}$ with

$$\tilde{\tilde{\tilde{G}}}\tilde{\gamma} = \bar{p},$$

and thus

$$G = \tilde{\tilde{\tilde{G}}}(I_L - \tilde{\gamma}\gamma^T),$$

and thus $\text{Rank}(G) = \text{Rank}(I_L - \tilde{\gamma}\gamma^T)$. If $\text{Rank}(I_L - \tilde{\gamma}\gamma^T) < L-1$, there are linearly independent vectors b, c with $\tilde{\gamma}\gamma^T b = b$ and $\tilde{\gamma}\gamma^T c = c$. This is impossible, since the columns of $\tilde{\gamma}\gamma^T$ are all proportional to $\tilde{\gamma}$, so $\text{Rank}(\tilde{\gamma}\gamma^T) \leq 1$. This completes the proof that $\text{Rank}(G) = L-1$.

We will now construct a matrix G^* with $G^*Gh = h$ for $h \perp \bar{p}$ and $G^*\bar{p} = 0$.

Observe that $\text{Col}(G) = \{b \in \mathbb{R}^L : b_L = 0\}$. Choose vectors $\{\tilde{d}_\ell\}_{\ell=1}^{L-1}$ that satisfy $G\tilde{d}_\ell = a_\ell$, and let

$$d_\ell \equiv \tilde{d}_\ell - (\tilde{d}_\ell \cdot \bar{p})\delta \ \ \forall \ell.$$

Because $d \cdot \bar{p} = 1$, $d_\ell \cdot \bar{p} = 0$; $d_\ell \perp \bar{p}\ \forall \ell$. Also, because $Gd_\ell = a_\ell\ \forall \ell$, the set $\{d_\ell\}$ is linearly independent. So $\{d_\ell\}$ spans the hyperplane orthogonal to $\bar{p}$.

If G^* is an $L \times L$ matrix whose ℓ^{th} column is d_ℓ for $\ell < L$, then by construction, $G^*Gh = h$ for $h \perp \bar{p}$. $G^*\bar{p} = 0$ is then be maintained by setting $G^*_{\ell L} = -\sum_{k=1}^{L-1} G_{\ell k}\bar{p}_k/\bar{p}_L$ for each ℓ.

By construction, we also have $\bar{p} \cdot G^* = \mathbb{0}_L^T$.

Choose a nonzero vector $\delta \in \text{Null}(G) \subset \text{Null}(M^T)$. Since $\bar{p} \notin \text{Col}(M)$ and the left null space is orthogonal to the column space, $\bar{p} \cdot \delta \neq 0$. Without loss of generality, choose δ such that $\bar{p} \cdot \delta = 1$.

Choose a production function $y(\cdot)$ that is $\mathcal{C}^1$ around $\bar{p}$, with $y(\bar{p}) = 0$ and $Dy(\bar{p}) = N$.

Choose an aggregate demand function $x(\cdot)$ that is $\mathcal{C}^2$ around $(\bar{p}, 0)$, with $x(\bar{p}, 0) = \bar{s}$ and $D_p x(\bar{p}, 0) = Ds(\bar{p}) - G^*$.

Choose $\bar{x}^I \gg 0, \ll \bar{s}$ and let $\bar{x}^{-I} \equiv \bar{s} - \bar{x}^I$. Let $\theta^I = 1$. Choose an increasing, strictly concave consumption utility function $v^I(\cdot)$ and an individual endowment e^I such that

- $x^I_{[0]}(\bar{p}, 0)$ $(\equiv \tilde{x}^I_{[0]}(\bar{p}, \bar{p} \cdot e^I)) = e^I$,
- $x^I_{[0]}(p, \pi)$ $(\equiv \tilde{x}^I_{[0]}(p, \bar{p} \cdot e^I + \pi))$ is defined and $\mathcal{C}^1$ in (p, π) in a neighborhood of $(\bar{p}, 0)$ (regarding $\tilde{x}^I_{[0]}(\cdot)$: $\mathcal{C}^1$ in p, B^i in a neighborhood of $(\bar{p}, \bar{p} \cdot e^I)$), and
- $\nabla_\pi x^I_{[0]}(\bar{p}, 0)$ $(\equiv \nabla_{B^I}\tilde{x}^I_{[0]}(\bar{p}, \bar{p} \cdot e^I)) = \delta$.

The wealth effect vector at specified prices $\bar{p}$ (here δ) and the Jacobian of demand with respect to prices at $\bar{p}$ can be set independently, subject to $\bar{p} \cdot \delta = 1$; see Mas-Colell et al. (1995), pp. 600–601.

Let $z^*(p) \equiv x(p, 0) - \bar{s}$ denote the excess demand function generated by aggregate demand function $x(\cdot)$ in an exchange economy with aggregate endowment vector $\bar{s}$. It is also the excess demand function generated by aggregate demand function $x(\cdot) - e^I$ in an exchange economy with aggregate endowment vector $\bar{s} - e^I$. Because $x(\cdot) - e^I$ is $\mathcal{C}^1$ in p around $(\bar{p}, 0)$, there is an $\epsilon > 0$ such that, within $\mathcal{N}_\epsilon(\bar{p})$, $z^*(\cdot)$ satisfies the second-to-last inequality near the bottom of p. 349 of Mantel (1974), with our p, $\bar{p}$, and $z^*_\ell(\cdot)$ (for each ℓ) taking the place of Mantel's q, p, and $g(\cdot)$.

Then by the theorem of Mantel (1974), there is a profile of $I-1$ continuous, increasing, strictly concave consumption utility functions $\{v^i(\cdot)\}$ and endowments $\{e^i\}$ with $e = \bar{s} - e^I$ implying the following demand functions, written as functions of prices and wealth:

$$\begin{aligned}\tilde{x}^i_{[0]}(p, B^i) &= \frac{B^i}{L}\text{Diag}(p)^{-1}\mathbb{1}_L - \frac{B^i}{\kappa^i}Dz^*(p)a_i, && i = 1, ..., L;\\ &= \frac{\kappa^i}{L}\left(\mathbb{1}_L - p\cdot\mathbb{1}_L\text{Diag}(p)^{-1}a_{i-L}\right) + \frac{B^i}{p_{i-L}}a_{i-L}, && i = L+1, ..., I-1\end{aligned}$$

for some nonzero constants $\{\kappa^i\}$, such that

$$\sum_{i=1}^{I-1}\tilde{x}^i_{[0]}(p, p\cdot e^i) - \bar{s} = z^*(p) \quad \text{for} \quad p \in \mathcal{N}_\epsilon(\bar{p}).$$

(Mantel notes only that his implied utility functions are monotone—i.e. weakly increasing—rather than strictly increasing; but because they are strictly quasiconcave throughout $\mathbb{R}^L_+$ (not just $\mathbb{R}^L_{++}$), they cannot be constant in any good throughout $\mathbb{R}^L_+$. Also, Mantel's construction consists of precisely $2L$ consumers; but given a construction for ν consumers, it can be extended to include $\nu+1$ consumers by "dividing" some i into collectively behaviorally equivalent j_1, j_2 with $e^j = e^i/2$ and $v^j(x^j) = v^i(2x^j)$ for $j \in \{j_1, j_2\}$.)

Given these endowments and profit shares $\theta^i = 0$ for $i < I$, we can write these individual demands as functions of prices and (trivially) profits:

$$x^i_{[0]}(p, \pi) = \frac{p\cdot e^i}{L}\text{Diag}(p)^{-1}\mathbb{1}_L - \frac{p\cdot e^i}{\kappa^i}Dz^*(p)a_\ell.$$

Since $z^*(p)$ is $\mathcal{C}^2$, $x^i_{[0]}(\cdot)$ is $\mathcal{C}^1$ in p, π for all $i < I$.

The continuous differentiability of $x_{[\mathbb{0}_{L\times I}]}(\cdot)$ and $y(\cdot)$, $\bar{p}\cdot\delta > 0$, and the implicit function theorem imply that the profit function $\pi(p, x_{[\mathbb{0}_{L\times I}]}(\cdot))$ is $\mathcal{C}^1$ in p around $\bar{p}$. So $\chi_{[\mathbb{0}_{L\times I}]}(\cdot)$ is $\mathcal{C}^1$ in p around $\bar{p}$. Also, $D\chi_{[\mathbb{0}_{L\times I}]}(\bar{p}) = D_p x(\bar{p}, 0)$, given $y(\bar{p}) = 0$ and the envelope theorem (which implies $\nabla_p\pi(p, x_{[\mathbb{0}_{L\times I}]}(\cdot)) = y(p)$). Summing the individual demands, $\chi_{[\mathbb{0}_{L\times I}]}(\bar{p}) = \bar{s} - e^I + e^I = \bar{s}$.

We now have an environment $(\{e^i\}, \{\theta^i\}, y(\cdot))$ and profile of consumption utility functions $\{v^i(\cdot)\}$ such that $s(\bar{p}) = \bar{s}$ and $Ds(\bar{p})G(\bar{p}, x_{[\mathbb{0}_{L\times I}]}(\cdot)) = M^T$. The latter equality holds because $Ds(\bar{p})G = M^T$, where by construction G equals the unique generalized inverse of $G^* = Ds(\bar{p}) - D\chi_{[\mathbb{0}_{L\times I}]}(\bar{p})$ with $G\nabla_\pi x_{[\mathbb{0}_{L\times I}]}(p, \pi) = G\delta = 0$ and a bottom row of zeroes; i.e. $G = G(\bar{p}, x_{[\mathbb{0}_{L\times I}]}(\cdot))$. It follows that

$$\psi(\bar{p}, x_{[\mathbb{0}_{L\times I}]}) = \psi.$$

Since for each i $v^i(\cdot)$ is strictly concave and increasing in all goods, its right derivative in each good ℓ (which we will denote $v^{i\prime}_\ell(\cdot)$) is everywhere defined and positive. Let $\mathcal{B}^i(p)$ denote i's budget set at prices $p \gg 0$, and choose an $\epsilon > 0$ and let

$$\mathcal{B}^i \equiv \{x^i : \bar{p} \cdot x^i \leq \bar{p} \cdot e^i + \epsilon\} \tag{B.70}$$

denote the budget set i faces given prices $\bar{p}$ under a relaxed budget constraint. Observe that $\mathcal{B}^i(p) \subset \mathcal{B}^i$ for all prices p near $\bar{p}$.

We will now show that for each i, for all $\bar{x}^i \in \mathcal{B}^i$, there is an $\epsilon^i(\bar{x}^i) > 0$ such that, for all $x^i \in \mathcal{N}_{\epsilon^i(\bar{x}^i)}(\bar{x}^i)$, $v^{i\prime}_\ell(x^i) > \underline{v}^{i\prime}_\ell$.

Let $\overline{\mathcal{B}}^i$ denote the upper boundary of $\mathcal{B}^i$: i.e. (B.70) with the weak inequality replaced with an equality. Fix i and $\bar{x}^i \in \overline{\mathcal{B}}^i$. The continuity and strict monotonicity of $v^i(\cdot)$ imply that there is an $r \gg 0$ and an indifference surface "IS_1", corresponding to consumption utility level $\bar{v}_1 > v^i(\bar{x}^i)$, that passes through $\bar{x}^i + r_\ell a_\ell$ for all ℓ. These properties also imply that IS_1 is bounded away from $\bar{x}^i$. There are thus also indifference surfaces "IS_2" and "IS_3", both bounded away from $\bar{x}^i$, corresponding to consumption utility levels $\bar{v}_2$, $\bar{v}_3$ with $v^i(\bar{x}^i) < \bar{v}_3 < \bar{v}_2 < \bar{v}_1$.

For some $\epsilon > 0$, there is a $\bar{r} \gg 0$ with $v^i(\bar{x}^i(1-\epsilon) + \bar{r}_\ell a_\ell) = \bar{v}_2$ for all ℓ. If for all $\epsilon > 0$ there were no such $\bar{r}$, then (by continuity and monotonicity of $v^i(\cdot)$) there would be an n-indexed sequence $\bar{x}^i(1-1/n) \to \bar{x}^i$ and an ℓ such that $v^i(\bar{x}^i(1-1/n) + r_\ell a_\ell) < \bar{v}_2$ for all n, even though by construction in the limit we have $v^i(\bar{x}^i(1-1/n) + r_\ell a_\ell) = \bar{v}_1 > \bar{v}_2$ (contradicting the continuity of $v^i(\cdot)$). Observe also that we must have $\bar{r} \gg \epsilon \bar{x}^i$, by monotonicity of $v^i(\cdot)$ and $\bar{v}_2 > v^i(\bar{x}^i)$. Choose such an ϵ and $\bar{r}$.

Let

$$\underline{x}^i \equiv \bar{x}^i(1-\epsilon)$$

and

$$X \equiv \bigtimes_{\ell=1}^{L} \left[\underline{x}^i_\ell,\ \underline{x}^i_\ell + \bar{r}_\ell\right].$$

That is, let X denote the box ranging from $\underline{x}^i_\ell$ to $\underline{x}^i_\ell + \bar{r}_\ell$ for each dimension ℓ. Observe that, given any point $x^i \in \text{IS}_3 \cap X$ and any ℓ, there is a unique and positive scalar $r_\ell(x^i) < \bar{r}_\ell$ such that $v^i(x^i + r_\ell(x^i)a_\ell) = \bar{v}_2$. (Existence follows from the continuity of $v^i(\cdot)$: consumption utility level $\bar{v}_2$ must be achievable by beginning with x^i and adding a quantity of good ℓ, since by monotonicity and $x^i > \underline{x}^i$ we have $v^i(x^i + \bar{r}_\ell a_\ell) > v^i(\underline{x}^i + \bar{r}_\ell a_\ell) = \bar{v}_2$.) By strict concavity, it follows that the right derivative of $v^i(\cdot)$ in good ℓ at x^i is greater than

$$\underline{v}^{i\prime}_\ell(\bar{x}^i) \equiv (\bar{v}_2 - \bar{v}_3)/\bar{r}_\ell > 0.$$

Then likewise, because IS_3 is bounded away from $\bar{x}^i$, there is an $\epsilon(\bar{x}^i) > 0$ such that $\mathcal{N}_{\epsilon^i(\bar{x}^i)}(\bar{x}^i) \cap \mathbb{R}^L_+ \subset X$ and, for all ℓ and $x^i \in \mathcal{N}_{\epsilon(\bar{x}^i)}(\bar{x}^i) \cap \mathbb{R}^L_+$, we have $x^i + \bar{r}_\ell(x^i)a_\ell \in$

$\text{IS}_3 \cap X$ for some $\tilde{r}_\ell(x^i) > 0$. By strict concavity, the right derivative of $v^i(\cdot)$ in good ℓ at this x^i is greater than $\underline{v}_\ell^{i\prime}(\bar{x}^i)$.

The set $\{\mathcal{N}_{\epsilon^i(\bar{x}^i)}(\bar{x}^i)\}_{\bar{x}^i \in \overline{\mathcal{B}}^i}$ constitutes an open cover of $\overline{\mathcal{B}}^i$. Because $\overline{\mathcal{B}}^i$ is compact, we can choose a finite subcover. There is thus for each ℓ a $\underline{v}_\ell^{i\prime} > 0$ such that the right derivative of $v^i(\cdot)$ in good ℓ at $\bar{x}^i$ is at least $\underline{v}_\ell^{i\prime}$ for all $\bar{x}^i \in \overline{\mathcal{B}}^i$.

Finally, by strict concavity, the $\{\underline{v}_\ell^{i\prime}\}$ lower-bound the right derivatives throughout $\mathcal{B}^i$.

Consider the consumption utility function profile $\{\tilde{v}^i(\cdot)\}$ with

$$\tilde{v}^i(x^i) = K^i v^i(x^i) - \psi^i \cdot x^i,$$

where

$$K^i = \max\left(1,\ \max_\ell -2\frac{\psi_\ell^i}{\underline{v}_\ell^{i\prime}}\right).$$

By construction, the right derivative of $\tilde{v}^i(\cdot)$ in each ℓ is positive throughout $\mathcal{B}^i$ for each i. Observe also that $\tilde{v}^i(\cdot)$ inherits the strict concavity of $v^i(\cdot)$: given any $\alpha \in [0,1]$, x^i, and $\tilde{x}^i$, we have

$$\begin{aligned}
&K^i v^i(\alpha x^i + (1-\alpha)\tilde{x}^i) - \psi^i \cdot (\alpha x^i + (1-\alpha)\tilde{x}^i) \\
&> \alpha K^i v^i(x^i) + (1-\alpha) K^i v^i(\tilde{x}^i) - \alpha \psi^i \cdot x^i - (1-\alpha)\psi^i \cdot \tilde{x}^i \\
&= \alpha\big(K^i v^i(x^i) - \psi^i \cdot x^i\big) + (1-\alpha)\big(K^i v^i(\tilde{x}^i) - \psi^i \cdot \tilde{x}^i\big).
\end{aligned}$$

Let $\tilde{\tilde{v}}^i(x^i) = \tilde{v}^i(x^i)$ for $x^i \in \mathcal{B}^i$ and let $\tilde{\tilde{v}}^i(\cdot)$ be increasing and strictly concave in all goods above the budget constraint, so that $\tilde{\tilde{v}}^i(\cdot)$ is globally increasing and strictly concave. Given consumption utility function profile $\{\tilde{\tilde{v}}^i(\cdot)\}$ and impact matrix ψ, for all i we have demands $\chi^i_{[\psi]}(p)$ equal to the quantities expressed above as $\chi^i_{[\mathbb{0}_{L\times I}]}(p)$ (with respect to $\{v^i(\cdot)\}$) throughout some neighborhood of $\bar{p}$. It follows that, given utility function profile $\{\tilde{\tilde{v}}^i(\cdot) + w^i(\cdot)\}$, $\psi(\bar{p}, \psi) = \psi$.