Growth given Cobb-Douglas automation

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1 Motivation

Aghion et al. (2019) §4.1, Example 3 states a proposition roughly identical to the proposition below, but its proof is incorrect. Davidson (2021) also states a proposition roughly identical to (in fact somewhat weaker than) the proposition below, but its proof is also incorrect. The aim of this document is to correct it.

2 Corrected proof

Following Aghion et al. (2019), let us say that a variable X exhibits a "Type II growth explosion" at some time t^* if $\lim_{t \uparrow t^*} X_t = \infty$.

Proposition 1 (Growth paths given Cobb-Douglas production and research).
Suppose

$$Y_t = A_t^{\sigma} K_t^{\alpha}, \tag{1}$$

$$\dot{K}_t = sY_t, \ \ \textit{and} \tag{2}$$

$$\dot{A}_t = A_t^{\phi} K_t^{\beta}, \tag{3}$$

where $A_0 > 0$, $K_0 > 0$, $\sigma > 0$, $\alpha \in (0,1)$, $s > 0$, $\phi < 1$, *and* $\beta > 0$, *and where (1)–(3) are defined for* $t \in [0, \infty)$*—or, if the system exhibits a Type II growth explosion at some time* t^**, for* $t \in [0, t^*)$.

*Thanks to Chad Jones for noting corrections to the following corrections!

Let

$$\gamma \triangleq \frac{\sigma}{1-\alpha}\frac{\beta}{1-\phi}. \tag{4}$$

If $\gamma > 1$, Y exhibits a Type II growth explosion.
If $\gamma = 1$, Y grows exponentially, with $\lim_{t\to\infty} g_{Y,t} = s^{\frac{\beta}{1+\beta-\alpha}}\left(\frac{\sigma}{1-\alpha}\right)^{\frac{1-\alpha}{1+\beta-\alpha}}$.
If $\gamma < 1$, Y grows power-functionally.

Proof. First observe that, for all t,

$$g_{Kt} = sA_t^{\sigma}K_t^{\alpha-1} \text{ and} \tag{5}$$

$$g_{At} = A_t^{\phi-1}K_t^{\beta}. \tag{6}$$

Let $\hat{g}_K$ ($\triangleq$"g_{g_K}") denote the proportional growth rate of g_K itself, and let $\hat{g}_A$ be defined likewise. It then follows from (5) and (6) that, for all t,

$$\hat{g}_{Kt} = \sigma g_{At} + (\alpha-1)g_{Kt} \text{ and} \tag{7}$$

$$\hat{g}_{At} = (\phi-1)g_{At} + \beta g_{Kt}. \tag{8}$$

If, for <u>any</u> time τ, $\hat{g}_{K\tau} > 0$ and $\hat{g}_{A\tau} > 0$, then

$$\sigma g_{A\tau} + (\alpha-1)g_{K\tau} > 0$$

$$\implies g_{A\tau} > \frac{1-\alpha}{\sigma}g_{K\tau}\ ; \tag{9}$$

$$(\phi-1)g_{A\tau} + \beta g_{K\tau} > 0$$

$$\implies g_{K\tau} > \frac{1-\phi}{\beta}g_{A\tau}\ ; \tag{10}$$

and thus

$$g_{A\tau} > \frac{1-\alpha}{\sigma}\frac{1-\phi}{\beta}g_{A\tau} \tag{11}$$

$$\implies \gamma > 1 \tag{12}$$

since $g_{A\tau} \geq 0\ \forall\tau$ by construction.

Likewise, if for any τ we have $\hat{g}_{K\tau} < 0$ $(= 0)$ and $\hat{g}_{A\tau} < 0$ $(= 0)$, then $\gamma < 1$ ($= 1$, respectively).

For any τ,

$$\hat{g}_{K\tau} = 0 \iff \hat{g}_{A\tau} = 0. \tag{13}$$

The "$\Rightarrow$" direction follows from (7). If $\hat{g}_{K\tau} = 0$, then $\sigma g_{A\tau} = (1-\alpha)g_{K\tau}$; so if the right-hand side is constant around τ, so is the left. The "$\Leftarrow$" direction follows likewise from (8).

Also, $\hat{g}_K$ and $\hat{g}_A$ are continuous in t wherever they are defined. So by the intermediate value theorem, if either term is negative at some time and positive at another time, it must equal zero at an intermediate time. By (13), we must then have $\gamma = 1$.

It follows that, if $\gamma \neq 1$, either

1. $\hat{g}_{Kt} > 0$ and $\hat{g}_{At} > 0$ $\forall t$,
2. $\hat{g}_{Kt} > 0$ and $\hat{g}_{At} < 0$ $\forall t$,
3. $\hat{g}_{Kt} < 0$ and $\hat{g}_{At} > 0$ $\forall t$, or
4. $\hat{g}_{Kt} < 0$ and $\hat{g}_{At} < 0$ $\forall t$,

with case 4 incompatible with $\gamma > 1$ and case 1 incompatible with $\gamma < 1$. We will now show that cases 2 and 3 are also incompatible with $\gamma \neq 1$.

Consider case 2. From $\hat{g}_{Kt} > 0$ $\forall t$, and (7), it follows that

$$g_{At} > \frac{1-\alpha}{\sigma} g_{Kt} \; \forall t. \tag{14}$$

Recall that, by stipulation, g_K always rising and g_A is always falling. Thus $\{g_{Kt}\}$ is bounded above, for instance by $\frac{\sigma}{1-\alpha} g_{A0}$, and $\{g_{At}\}$ is bounded below, for instance by $\frac{1-\alpha}{\sigma} g_{K0}$. By the monotone convergence theorem for functions, $\lim_{t\to\infty} g_{Kt}$ and $\lim_{t\to\infty} g_{At}$ are defined (and finite, and—since $g_{K0} = sA_0^\sigma K_0^{\alpha-1} > 0$—positive). Let us denote these limits g_K^* and g_A^* respectively.

By (7) and (8), it then follows that $\lim_{t\to\infty} \hat{g}_{Kt}$ and $\lim_{t\to\infty} \hat{g}_{At}$ are also defined (and finite). Since g_K^* and g_A^* are finite and nonzero, as we have just shown, it must be that $\lim_{t\to\infty} \hat{g}_{Kt} = \lim_{t\to\infty} \hat{g}_{At} = 0$. Taking the limits of terms (7) and (8), we then have

$$g_A^* = \frac{1-\alpha}{\sigma} g_K^* \text{ and} \tag{15}$$

$$g_K^* = \frac{1-\phi}{\beta} g_A^*, \tag{16}$$

which jointly imply $g_A^* = \gamma g_A^*$ and thus $\gamma = 1$.

Case 3 can be shown to imply $\gamma = 1$ by a precisely analogous proof. Thus $\gamma > 1$ implies case 1 and $\gamma < 1$ implies case 4.

Suppose $\gamma > 1$. By the statements of case 1 and expressions (7)–(8), we have

$$g_{At} > \frac{1-\alpha}{\sigma} g_{Kt} \ \forall t \text{ and} \tag{17}$$

$$g_{Kt} > \frac{1-\phi}{\beta} g_{At} \ \forall t. \tag{18}$$

By (17), and substituting by expressions (5) and (6),

$$\begin{aligned} g_{At}^2 &> \frac{1-\alpha}{\sigma} g_{At}\, g_{Kt} \\ &= s\frac{1-\alpha}{\sigma} A_t^{\phi-1+\sigma} K_t^{\beta+\alpha-1} \ \forall t. \end{aligned} \tag{19}$$

If the relationship of (17) were an equality at all t, then A would always grow at precisely the same proportional rate as $K^{\frac{1-\alpha}{\sigma}}$. Noting that

$$A_0 = A_0 K_0^{-\frac{1-\alpha}{\sigma}} \cdot K_0^{\frac{1-\alpha}{\sigma}}, \tag{20}$$

we would maintain this ratio between A and $K^{\frac{1-\alpha}{\sigma}}$, with

$$A_t = A_0 K_0^{-\frac{1-\alpha}{\sigma}} K_t^{\frac{1-\alpha}{\sigma}} \ \forall t. \tag{21}$$

It thus follows from (17) that

$$A_t \geq A_0 K_0^{-\frac{1-\alpha}{\sigma}} K_t^{\frac{1-\alpha}{\sigma}} \ \forall t \tag{22}$$

$$\implies K_t \leq K_0 A_0^{-\frac{1-\alpha}{\sigma}} A_t^{\frac{\sigma}{1-\alpha}} \ \forall t \tag{23}$$

(with equality at $t = 0$ and strict inequality at $t > 0$). It likewise follows from (18) that

$$K_t \geq K_0 A_0^{-\frac{1-\phi}{\beta}} A_t^{\frac{1-\phi}{\beta}} \ \forall t. \tag{24}$$

So, if $\beta + \alpha - 1 \leq 0$, it follows from (19) and (23) that

$$g_{At}^2 > s\frac{1-\alpha}{\sigma} A_0^{-\frac{\sigma}{1-\alpha}(\beta+\alpha-1)} K_0^{\beta+\alpha-1} A_t^{\phi-1+\sigma+\frac{\sigma}{1-\alpha}(\beta+\alpha-1)} \ \forall t. \tag{25}$$

Given $\gamma > 1$, the exponent on A_t in (25) is positive. Likewise, if $\beta+\alpha-1 > 0$, it follows from (19) and (24) that

$$g_{At}^2 > s\frac{1-\alpha}{\sigma}A_0^{-\frac{1-\phi}{\beta}(\beta+\alpha-1)}K_0^{\beta+\alpha-1}A_t^{\phi-1+\sigma+\frac{1-\phi}{\beta}(\beta+\alpha-1)}\ \forall t. \tag{26}$$

Again, given $\gamma > 1$, the exponent on A_t in (26) is positive. Either way, therefore, A grows at worst hyperbolically, and so exhibits a Type II growth explosion. It follows immediately that Y does as well.

If $\gamma < 1$, a proof that A grows at best power-functionally is precisely analogous, except that it uses inequality (24) in the $\beta + \alpha - 1 \leq 0$ case and inequality (23) in the $\beta + \alpha - 1 > 0$ case. By (8) and the case 4 stipulation that $\hat{g}_{At} < 0\ \forall t$, we then have

$$g_{Kt} < \frac{1-\phi}{\beta}g_{At}\ \forall t, \tag{27}$$

implying that K also grows at best power-functionally. Thus Y grows at best power-functionally as well.

Furthermore, it follows from (3) that if K were constant, A (and thus Y) would grow power-functionally. Since the possibility of capital accumulation cannot decelerate output growth, Y does in fact grow power-functionally.

Let us last consider the case of $\gamma = 1$.

From (7), we know that if $g_{Kt} > \frac{\sigma}{1-\alpha}g_{At}$ then $\hat{g}_{Kt} < 0$, and vice-versa. Likewise, from (8), we know that if $g_{At} > \frac{\beta}{1-\phi}g_{At}$ then $\hat{g}_{At} < 0$, and vice-versa. When $\gamma = 1$, however,

$$\frac{\beta}{1-\phi} = \frac{1-\alpha}{\sigma}. \tag{28}$$

It follows that

$$g_{K0} \geq \frac{\sigma}{1-\alpha}g_{A0} \tag{29}$$

$$\iff g_{Kt} \geq \frac{\sigma}{1-\alpha}g_{At}\ \forall t. \tag{30}$$

By the reasoning following (14), the limits $g_K^* \triangleq \lim_{t\to\infty} g_{Kt}$ and $g_A^* \triangleq \lim_{t\to\infty} g_{At}$ are defined, finite, and positive. Furthermore, by the continuity of $\hat{g}_K$ and $\hat{g}_A$ in g_K and g_A, we must have $g_K^* = \frac{\sigma}{1-\alpha} g_A^*$. Thus

$$\frac{g_K^*}{g_A^*} = \frac{\sigma}{1-\alpha} \tag{31}$$

$$\Longrightarrow \lim_{t\to\infty} s A_t^{\sigma+1-\phi} K_t^{\alpha-1-\beta} = \frac{\sigma}{1-\alpha} \tag{32}$$

$$\Longrightarrow \lim_{t\to\infty} A_t^{\sigma} K_t^{\alpha-1} = \left(\frac{\sigma}{s(1-\alpha)}\right)^{\frac{1-\alpha}{1+\beta-\alpha}} \qquad \text{by } \gamma = 1 \tag{33}$$

$$\Longrightarrow g_K^* = s^{\frac{\beta}{1+\beta-\alpha}} \left(\frac{\sigma}{1-\alpha}\right)^{\frac{1-\alpha}{1+\beta-\alpha}} \quad \text{by (5).} \tag{34}$$

Finally,

$$\lim_{t\to\infty} g_{Yt} = \sigma g_A^* + \alpha g_K^* \tag{35}$$

$$= \sigma \frac{1-\alpha}{\sigma} g_K^* + \alpha g_K^* \qquad \text{by (31)} \tag{36}$$

$$= g_K^*. \tag{37}$$

□