

Ethical Consumerism

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September 22, 2023

Abstract

I study a static production economy in which consumers have not only preferences over their own consumption but also preferences—e.g. ethical preferences—over the aggregate supply of each good. Though existing work on the implications of such preferences assumes that consumers act as price-takers, I show that consumers with such preferences generically choose not to act even approximately as price-takers, when permitted to act strategically. I therefore introduce a near-Nash equilibrium concept that generalizes the near-Nash equilibria found in literature on strategic foundations of general equilibrium to the case of where consumers care about both consumption and supply. I also find [currently narrow] sufficient criteria under which such equilibria exist, and closed-form characterizations of consumer behavior in all such equilibria.

1 Introduction

1.1 Motivation

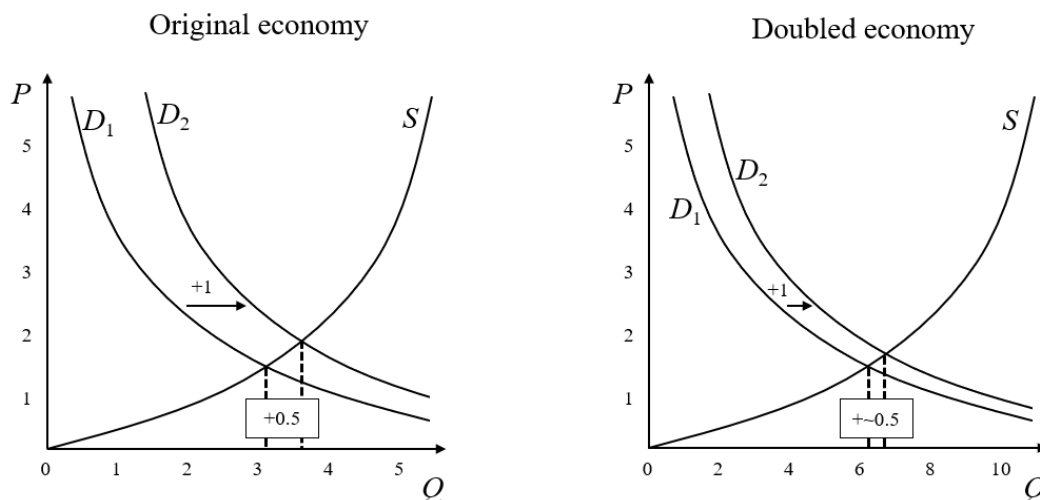
A consumer’s purchasing behavior affects not only the quantities of goods she herself consumes, but also the quantities others consume and the total quantities of each good supplied. She may have preferences over all these quantities, and she may, as a result, optimize her purchasing behavior accordingly. This paper explores a model of general equilibrium in which consumers do so.

Consider, for example, a consumer with concern for animal welfare. The consumer’s utility is increasing in his own meat consumption, holding supply fixed, but decreasing in total meat supply: in the supply of pigs and chickens in particular. The latter effect may, and often does, motivate consumers to purchase less meat than they would otherwise (or none at all).

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It bears emphasizing that, despite the well-known result (Roberts and Postlewaite, 1976) that a consumer's ability to impact equilibrium prices generically vanishes as the economy grows large, her ability to impact equilibrium supply does not. This is because, as the economy grows large and the price-impacts of an individual consumer's demand behavior shrink to zero, any such price-impact influences the purchases of a number of other consumers that rises to infinity.

Consider a consumer's decision to buy one more unit of some good at any given price. Compare (a) the impact of this decision on that good's equilibrium prices and supply levels in a given economy with (b) the impact of the decision on equilibrium prices and supply levels in a "doubled" economy with twice as many agents but an identical distribution of endowments, preferences, profit shares, and production technologies. In the doubled economy, quantities supplied and demanded at any given price will double:



In the doubled economy, the units on the vertical axis do not change, but the units on the horizontal axis are doubled, as the rightward movement of the demand curve on the page resulting from a one-unit decrease in demand is halved. Thus, if buying one unit of some good causes its production to increase by 0.5 in an economy with, say, one billion participants, it also causes its production to increase by approximately 0.5 in the economy with two billion participants. The size of an "ethical externality" in this sense does not in general fall to zero as price impacts fall to zero and an economy approaches perfect competition.

In light of these ethical externalities, how should an animal-welfare-conscious consumer should adjust his demands, relative to what they would be if he had no concern for animal welfare? If he believes that the production of a dollar's worth of chicken creates more misery than the production of a dollar's worth of pork, he may naively be inclined to prioritize reducing his purchases of chicken over reducing his purchases

of pork. If the supply of pork is more price-elastic and demand for pork less price-elastic than that of chicken, however, this inclination may be misguided. Buying less chicken in this case simply causes the price to fall and the quantity demanded by other consumers to rise, with little net impact on the quantity of chicken consumed. Buying less pork, by contrast, generates a substantial decrease to the quantity of pork consumed. Cutting back on pork may thus be the higher priority.

Complicating matters further, our consumer must consider the impact of his purchases of a good not only on the quantity of that good, but on the quantities of all the goods he cares about. If buying less chicken causes other consumers to substitute to chicken from other meat products, whereas buying less pork causes other consumers to substitute to pork from vegetables, then cutting back on chicken may be the best policy after all.

Though this paper is intended primarily as a model of ethical consumerism, and the precise modeling assumptions made will be tailored to the ethical consumerist context, it is worth noting that other agents with preferences over total supply levels also face the motivations and challenges described above. Suppose, for instance, that some goods impose more conventional (i.e. not “ethical”) externalities on a consumer, and that these externalities depend on the goods’ absolute supply levels rather than on supply per person. Then the utility-impacts of the consumer’s contributions to supply are not close to zero even when her proportional contributions are small. When she decides what to buy, she too must consider the impacts of her purchases on the equilibrium supply levels of all the goods that impose externalities on her.

This paper aims to characterize, in light of these complications, equilibrium market behavior by “ethical consumers”—and other consumers with preferences over own good-consumption levels, others’ good-consumption levels, and total supply levels—in a competitive production economy. That is, we will study strategic consumer behavior in general equilibrium with externalities.

1.2 Related literature

Though this appears to be the first model to incorporate all three features (strategic behavior, general equilibrium, and externalities), there exist literatures exploring all three pairs of these features in isolation.

First, there is an extensive literature exploring strategic behavior in general equilibrium without externalities. A central concern of this literature is to put the standard Walrasian model—in which agents are forbidden from strategic behavior and must act like perfect price-takers—on a strategic footing. This is done by constructing a game in which firms and/or individuals can choose their supply or demand correspondences, respectively, and equilibrium prices set the excess demands implied by these chosen correspondences equal to zero. Consider a sequence of such games set

in ever larger “replicated economies”, with each agent becoming an ever smaller part of the whole, and consider the sequence of Nash equilibria of these games. We can then ask under what circumstances, and in what sense, these Nash equilibria—these profiles of chosen supply and demand correspondences—can or must converge to the “Walrasian” profile in which agents all choose their price-taking supply or demand correspondences.

Roberts and Postlewaite (1976) show that, in an exchange economy, the demand correspondences chosen in a sequence of Nash equilibria do not generally converge, even pointwise, to the Walrasian demand correspondences. They do find, however, that a consumer’s impacts on equilibrium prices, equilibrium allocations, and her own utility by choosing her Walrasian demand correspondence fall to zero as the economy is replicated. It follows that if a consumer faces arbitrarily small costs—e.g. computational costs—to deviating from price-taking behavior, she will choose to act as a price-taker in a sufficiently large economy.

Otani and Sicilian (1990), studying a restriction to the same game, demonstrate that if consumers can only choose differentiable demand functions, there are sequences of Nash equilibria that, again, converge to Walrasian equilibria. Jackson and Manelli (1997) find that uncertainty about others’ chosen strategies can also motivate consumers to adopt behavior that converges to fully price-taking behavior.

Though these papers all discuss exchange economies, they are perhaps the papers on strategic foundations for general equilibrium most relevant to this one, which will feature a production economy but strategic behavior only by consumers. Nevertheless, the literature contains numerous other convergence results, including results set in production economies featuring strategic behavior by producers. The message generally taken from this literature appears to be that Walrasian equilibrium reasonably approximates what one should expect to obtain in an economy in which prices result from the strategic behavior of consumers and/or firms. As we will see, and as suggested by the informal discussion of §1.1, the presence of consumers with ethical preferences, or supply externalities more generally, allows no such approximation.

A second literature explores general equilibrium with externalities in settings in which individuals are not permitted to act strategically.

Dufwenberg et al. (2011), for instance, present a model intended to capture equilibrium purchasing behavior in a production economy in which individuals may have “other-regarding preferences”: preferences defined not only over their own consumption baskets but also over the consumption baskets, utility levels, or budget sets achieved by other consumers. If Dufwenberg et al. had allowed consumers to choose their demand functions strategically, their project would have encompassed the project of the present paper, since concern for the total production of various goods is a special case of concern for others’ consumption baskets. Instead, however, Dufwenberg et al. maintain the classical assumption that each individual is *fully* price-taking, in the sense that she acts as if her own purchases have no impact on

prices and thus no impact on aggregate production. They thus conclude that an individual’s other-regarding preferences have no impact on her purchasing behavior unless they are not separable from her self-regarding preferences: for example, if she prefers brown bread to white bread if and only if her neighbor has two cars.

A similar “fully price-taking” approach to other-regarding preferences, or preferences over total production levels, is taken by Kreps (1990) (p. 203), Ellickson (1994) (§7.3), Sobel (2009), and others.

The approach taken in this literature may be motivated by the observation from the literature on strategic foundations for general equilibrium that, in large economies and in the absence of externalities, relatively mild assumptions can guarantee that strategic behavior differs little (or produces outcomes that differ little) from price-taking behavior. In contrast, the present paper is centered on the insight that, as noted above, externalities such as ethical preferences will typically motivate strategic consumer behavior that differs substantially from price-taking behavior, however large the economy relative to each consumer.

Finally, at least two strands of literature explore the implications of strategic demand behavior by individuals with preferences over supply levels, but do so in a partial equilibrium setting.

The first is the general literature on the private provision of public goods. Bergstrom et al. (1986) study equilibrium spending behavior by public good providers in light of crowd-out issues like those discussed above. The model we will consider is thus conceptually related to the model introduced by Bergstrom et al. and developed throughout the subsequent literature on the private provision of public goods and bads.

Our model, however, will vary from the models typically explored by that literature in three interrelated ways. First, whereas models of public good provision games typically treat prices as exogenous, we will allow goods’ prices to be determined endogenously by the starting endowments, production technologies, and quantities of other goods purchased. Second, we will allow some individuals to have preferences that depend only on their own consumption baskets and not at all on total supply levels. (Without the first variation—i.e. the introduction of endogenous prices—fully “selfish” individuals have no impact on the game played among those who do care about total supply levels. Selfish individuals can therefore safely be excluded from the model.) Third, we will assume that all agents are small, in the sense that they can act as price-takers for some purposes.

The second is the literature, apparently confined so far entirely to agricultural economics, on “equilibrium displacement models”, or EDMs. (See Wohlgenant (2011) for a review.) EDMs generate predictions about the impact that purchases of a particular good have on that good’s equilibrium supply level. Norwood and Lusk (2011), for instance, estimate the price elasticities of supply and demand for various animal products and, from these estimations, calculate the extent to which marginal

purchases of a given animal product change the equilibrium quantity supplied of that product. Wilkinson (2022) uses a similar EDM analysis in arguing that consumers are morally obligated to account for the market externalities of their purchases: i.e. the impacts that their purchases have on prices, and the impacts of these price-shifts on others' purchases and ultimately others' welfare.

The two pieces cited above—along with most of the EDM literature—only consider the impact of purchasing a certain good on the equilibrium supply of that good. In effect, they consider only the equilibrium supply shift presented graphically in §1.1. For our purposes, such an analysis is relevant only in the event that cross-price elasticities of demand and supply are zero across goods over whose supply a consumer has ethical preferences. Goods about which one has ethical preferences, however, are often highly substitutable (for consumers and other producers) with goods about which one has similar ethical preferences. Recall the example of §1.1: a consumer who reduces her consumption of chicken on ethical grounds is likely to be concerned with decreasing the production not only of chicken but also of other meat products. Partial equilibrium analyses like the two cited above may thus be highly misleading.

An obscure but especially relevant step toward generality, within the EDM literature, is developed by Gardner (1987). Gardner's framework allows for an analysis of multiple goods simultaneously, in which changes to an individual's demand for one good affect the equilibrium prices and supply levels of other goods under consideration. His framework nonetheless falls short of a general equilibrium framework in two ways. First, it can serve only as a partial guide to the individual ethical consumer, as it breaks down when the consumer attempts to account for her impacts on all markets at once. Second, it serves *only* as a guide to the individual ethical consumer. It does not address the complexities in defining and finding the equilibria that arise when multiple (let alone all) consumers choose demands in light of their ethical preferences. It is thus in no way continuous with the literature on public good games, referenced above.

The general equilibrium analysis presented here fills this gap.

2 Model

2.1 Overview

There are $L \geq 2$ goods.

The production technology allows a production vector $y \in \mathbb{R}^L$ if $F(y) \leq 0$, where $F(\cdot) : \mathbb{R}^L \rightarrow \mathbb{R}$ is a strictly increasing, strictly convex, \mathcal{C}^1 function with $F(\mathbf{0}_L) = 0$.

There are I individuals, with starting endowments $\omega^i \geq 0$ (or $\omega_\ell^i \geq 0$ of good ℓ) and utility functions u^i for $i = 1, \dots, I$. The vector of total initial endowments is ω , and the total initial endowment of good ℓ is ω_ℓ . Likewise, the vector of net

production quantities is y , and the net production of good ℓ is y_ℓ .

Supply is denoted $s \triangleq \omega + y$. Supply cannot be negative. It follows from our assumptions on $F(\cdot)$ that the production possibility set

$$Y \triangleq \{y : F(y) \leq 0, y_\ell \geq -\omega_\ell \forall \ell\} \quad (1)$$

is compact. (See Mas-Colell et al. (1995), Proposition 16.AA.1.)

Given any prices $p \gg 0$, production y is chosen to maximize $p \cdot y$:

$$y(p) \triangleq \operatorname{argmax}_{y \in Y} p \cdot y. \quad (2)$$

It follows from our assumptions that $y(p)$ is single-valued.¹ Of course, since the value that maximizes a function also maximizes any monotonic transformation of it, $y(p)$ and $s(p) (\triangleq \omega + y(p))$ are homogeneous of degree zero.

Given $y : F(y) = 0$, the marginal rates of transformation between goods are given by the ratios of entries in $\nabla F(y)$ ($\gg 0$). Observe that

$$\bar{Y} \triangleq \{y \in Y : F(y) = 0\} \quad (3)$$

is compact: it is bounded because it is a subset of Y , which is bounded, and it is closed because it is the preimage of the continuous function $F(\cdot)$ on the closed set $\{0\}$. It then follows from the continuous differentiability of $F(\cdot)$ that

$$P \triangleq \{\nabla F(y) / (\nabla F(y))_L\}_{y \in \bar{Y}} \quad (4)$$

is compact. Intuitively, P is the set of price vectors for which $y(p)$ is not constrained by the endowment.

Let

$$\tilde{P} \triangleq \{p : s(p) \gg 0, p_L = 1\}. \quad (5)$$

$\tilde{P} \subset P$. By contradiction, consider a $p \in \tilde{P}$. If $p \notin P$, there exist ℓ, k such that $\bar{p}_\ell / \bar{p}_k > (\nabla F(y(\bar{p})))_\ell / (\nabla F(y(\bar{p})))_k$. Small Δy with $\Delta y_\ell > 0$, $\Delta y_k < 0$,

$$-\frac{\Delta y_k}{\Delta y_\ell} \in \left(\frac{(\nabla F(y(\bar{p})))_\ell}{(\nabla F(y(\bar{p})))_k}, \frac{\bar{p}_\ell}{\bar{p}_k} \right), \quad (6)$$

and $\Delta y_m = 0$ ($m \neq \ell, k$) are thus feasible, by $s(\bar{p}) \gg 0$, and strictly profit-increasing.

We will assume that $F(\cdot)$ is such that

$$\bar{y}(p) \triangleq \operatorname{argmax}_{y : F(y) \leq 0} p \cdot y \quad (7)$$

¹Because Y is compact, a maximum exists. Because $F(\cdot)$ is strictly convex, the maximum is unique: a convex combination of two maxima would earn equal profits, and allow for further profit-increasing production.

is \mathcal{C}^1 on P , and therefore that $y(p)$ is \mathcal{C}^1 throughout \tilde{P} (with $J_y(p) = J_{\bar{y}}(p)$ for $p \in \tilde{P}$).

Aggregate profits are denoted π .² Profits are divided among consumers according to shares $\theta^i \geq 0$. Consumer i 's budget thus equals $p \cdot \omega^i + \theta^i \pi$.

Individual i 's demand function is denoted $x^i(\cdot)$, and a vector of purchases by i is denoted x^i . The aggregate demand function, or sum of individual demand functions, is denoted $x(\cdot)$, and the vector of total purchases of each good is denoted x .

Individual i has a utility function $u^i(\cdot)$ representing her all-things-considered preferences over $(x^i, s) \in \mathbb{R}_{\geq 0}^{2L}$. We will assume that her preferences are additively separable across x^i and s , with her utility function taking the form

$$u^i(x^i, s) = v^i(x^i) + w^i(s). \quad (8)$$

Her consumption utility function $v^i(\cdot)$ is strictly increasing and either (a) strictly concave or (b) quasilinear in good L and strictly concave in goods $\ell < L$. Her externality function $w^i(\cdot)$ is³ CRS and \mathcal{C}^1 , noting in particular that $\nabla w^i(s)$ is finite at all $s \geq 0_L$.

We will assume that there is at least one individual whose preferences are defined only over own consumption and who is endowed with positive endowments and a positive profit share. We will denote some such individual I . More formally,

$$w^I(s) = 0 \quad \forall s, \quad (9)$$

$$\theta^I > 0, \quad (10)$$

$$\omega^I \gg 0. \quad (11)$$

Prices, profits, and production levels are the outcome of a game that unfolds as follows. First, each individual i chooses a demand function $x^i(p, \pi)$ that is feasible, continuous, and homogeneous of degree zero, and that maintains $p \cdot x^i(p, \pi)$ nondecreasing in π . Second, normalized prices p^* and a profit rate π^* are found that clear all markets (as will be shown to exist). Third, production y is chosen to maximize $p^* \cdot y$, and each consumer i receives his demanded basket $x^i(p^*, \pi^*)$.

We will assume for now that I sets $x^I(\cdot)$ equal to his price- (and profit-)taking demand function, noting that this function exists and is unique by the implied strict quasiconcavity of $v^I(\cdot)$. Propositions 4 and 5 will justify this assumption in

²If consumers exhaust their budgets, aggregate profits equal $p \cdot y(p)$. Though we will assume that production is profit-maximizing, we will not in general assume that consumers exhaust their budgets. Nevertheless, as shown below, the production vector y that maximizes $p \cdot y$ will also maximize profits. Aggregate production is thus profit-maximizing. The definition of aggregate profits more generally is given by (12) below.

³These terms are intended only as concise ways to distinguish $v^i(\cdot)$, used to represent the preferences i has over the quantities of goods she herself purchases, from $w^i(\cdot)$, used to represent the preferences i has over total supply levels. The latter will often consist primarily of concerns about the externalities that supply imposes on others, but as noted in §1, it need not do so.

equilibrium.

Given positive prices and an aggregate demand function, aggregate profits equal

$$\pi(p, x(\cdot)) \triangleq \underline{\pi \text{ such that } p \cdot x(p, \pi) = p \cdot (\omega + y(p))}. \quad (12)$$

Note that

$$p \cdot x(p, p \cdot y(p)) \leq p \cdot (\omega + y(p)) \quad (13)$$

by the feasibility of the chosen demand functions. Also, because the chosen demand functions $x^i(\cdot)$ are continuous and maintain $p \cdot x^i(p, \pi)$ nondecreasing in π , we have that $p \cdot x(p, \pi)$ is continuous and nondecreasing in π . Finally, by (9), (10), the assumption that $v^I(\cdot)$ is strictly increasing, and the assumption that I chooses his price-taking demand function, $p \cdot x(p, \pi)$ is in fact strictly increasing in π , with

$$\lim_{\pi \rightarrow \infty} p \cdot x(p, \pi) = \infty. \quad (14)$$

Thus (12) is well-defined: there is always exactly one π such that $p \cdot x(p, \pi) = p \cdot (\omega + y(p))$. Furthermore, this π will be greater than or equal to $p \cdot y(p)$.

This definition of aggregate profits is a standard alternative to the more conventional $\pi(p, y) = p \cdot y$ used in the event that individuals are not locally nonsatiated.⁴ Here, we do not assume local nonsatiation: we allow for the possibility that, in some circumstances and for some preferences individuals might have about aggregate supply levels, an individual maximizes his utility by choosing a demand function in which he does not always exhaust his budget. Defining profit to equal $p \cdot y$ would imply that, even in equilibrium, more-than-satiated consumers would give rise to $p \cdot x < p \cdot s$. Firms would in effect remain in possession of unsold goods of value of $p \cdot (s - x)$, leaving them to rot on the shelves.

Definition (12), by contrast, amounts to the more natural and realistic assumption that any such excess production would belong to the producer, and thus to individuals according to shares θ . This implies that if $p \cdot x(p, \pi) < p \cdot s(p)$, profits π are incompatible with $(p, x(\cdot))$. Given p , aggregate profits in fact exceed π by $p \cdot (s - x)$. Likewise, if $p \cdot x(p, \pi) > p \cdot s(p)$, profits π are incompatible with $(p, x(\cdot))$ because they imply that consumers buy goods worth more than all the goods available to sell. In this case at least some of the dividends returned to consumers were not true profits at all; firms would have had to withhold at least some of these profits in order to fill the orders given by $x(\cdot)$.

Observe that when individuals do exhaust their budgets,

$$p \cdot x(p, p \cdot y) = p \cdot \omega + p \cdot y = p \cdot s. \quad (15)$$

⁴Kononov (2005) offers an review of the literature using this approach, at least as of 2005. The resulting equilibria are called “dividend equilibria” or “Walrasian equilibria with slack”.

Profits then equal $p \cdot y$, as usual. Also, even in the more general setting of (12), because $p \cdot x(p, \pi)$ is increasing in π , assuming that production $y(p)$ maximizes profits is equivalent to assuming that production maximizes $p \cdot s$ and thus $p \cdot y$.

Let us make two clarifications.

First, it may seem as though individuals consider their impact on total production only in their purchasing choices and not in their selling choices. That is, it may seem as though individuals are assumed to maximize their income by selling the entirety of their endowments, regardless of the extent to which their endowments are used in the production of goods they consider to have negative externalities. In fact, however, refusing to sell part of one's endowment ω_ℓ^i is here equivalent to increasing one's purchases of x_ℓ . If such purchases would produce a favorable marginal effect on s , in equilibrium, then i will set $x^i(\cdot)$ so as to motivate these purchases in equilibrium.

Second, note that instead of introducing a set of profit-maximizing firms, we are simply positing that production maximizes aggregate profits subject to a single, abstract production technology. If we modeled the firms explicitly, we would have to model how their behavior responded to the diverse preferences of their owners, not all of whom here care only about profit maximization. We would likewise have to explore the possibility that some distributions of firm ownership are incompatible with equilibrium. Individuals with preferences regarding the production levels of some goods might, for instance, want to concentrate their capital holdings in a few relevant firms so as to wield influence as “activist investors”. To restrict our focus to the equilibrium consequences of market purchases, we will therefore simply assume that production proceeds along profit-maximizing lines.

2.2 Formal summary

Definition 1. An economy \mathcal{E} is a tuple of endowments $\{\omega^i\}_{i=1}^I$, utility functions $\{u^i(\cdot)\}_{i=1}^I$, profit shares $\{\theta^i\}_{i=1}^I$, and a production function $y(\cdot)$ all compatible with

the assumptions of the model introduced in §2.1: namely that

$$\omega^i \geq 0 \quad \forall i, \quad \omega^I \gg 0; \quad (16)$$

$$u^i(x^i, s) = v^i(x^i) + w^i(s), \text{ where both } v^i(\cdot) \text{ and } w^i(\cdot) \text{ are defined on } \mathbb{R}_{\geq 0}^L, \quad (17)$$

$v^i(\cdot)$ is strictly increasing and either (a) strictly concave or
(b) quasilinear in good L and strictly concave in goods $\ell < L$,
and $w^i(\cdot)$ is CRS and \mathcal{C}^1 ;

$$w^I(\cdot) = 0; \quad (18)$$

$$\theta^i \geq 0 \quad \forall i, \quad \theta^I > 0; \quad (19)$$

$$\text{and } y(p), \text{ defined on } \mathbb{R}_{>0}^L, \text{ maximizes } p \cdot y \text{ given } y \in Y, \text{ where} \quad (20)$$

$Y \triangleq \{y : y \geq -\omega \text{ and } F(y) \leq 0\}$ for some strictly increasing,
strictly convex, \mathcal{C}^1 function $F(\cdot)$ with $F(0_L) = 0$ and

$$\bar{y}(p) \triangleq \max_{y: F(y)=0} p \cdot y \text{ defined and } \mathcal{C}^1 \text{ for } p \in \{\nabla F(y)\}_{y: F(y)=0}.$$

Definition 2. A function $x^i(p, \pi)$ from $\mathbb{R}_{>0}^L \times \mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}^L$ is an admissible demand function for $i < I$ in economy \mathcal{E} if $x^i(\cdot)$ is (a) continuous, (b) h.o.d. 0, and (c) feasible in the sense that

$$p \cdot x^i(p, \pi) \leq p \cdot \omega^i + \theta^i \pi \quad \forall (p, \pi) \in \mathbb{R}_{>0}^L \times \mathbb{R}_{\geq 0}, \quad (21)$$

and (d) if $p \cdot x^i(p, \pi)$ is nondecreasing in π .

An admissible demand function profile in \mathcal{E} is a demand function profile $\{x^i(\cdot)\}_{i=1}^I$ for which $x^i(\cdot)$ satisfies (a)–(d) above for all $i < I$ and $x^I(\cdot)$ is I 's price- (and profit-)taking demand function.

An admissible aggregate demand function in \mathcal{E} is the sum over i of the elements $\{x^i(\cdot)\}$ of an admissible demand function profile in \mathcal{E} .

We will denote the set of admissible demand functions for i in \mathcal{E} by $\mathbb{A}^i(\mathcal{E})$, the set of admissible demand function profiles in \mathcal{E} by $\{\mathbb{A}^i(\mathcal{E})\}$, and the set of admissible aggregate demand functions in \mathcal{E} by $\mathbb{A}(\mathcal{E})$.

Definition 3. Given economy

$$\mathcal{E} = (\{\omega^i\}_{i=1}^I, \{u^i(x^i, s)\}_{i=1}^I, \{\theta^i\}_{i=1}^I, y(p)), \quad (22)$$

the n -replicated economy equals

$$\mathcal{E}^{(n)} \triangleq (\{\omega^{i \bmod I}\}_{i=1}^{nI}, \{u^{i \bmod I}(x^i, s)\}_{i=1}^{nI}, \{\theta^i/n\}_{i=1}^{nI}, ny(p)). \quad (23)$$

n-replicated demand functions of various kinds equal

$$x^{i(n)}(p, \pi) \triangleq x^i(p, \pi/n), \quad (24)$$

$$x^{(n)}(p, \pi) \triangleq nx(p, \pi/n), \quad (25)$$

$$x^{-i(n)}(p, \pi) \triangleq x^{(n)}(p, \pi) - x^{i(n)}(p, \pi), \text{ and} \quad (26)$$

$$\{x^i(p, \pi)\}_{i=1}^I \triangleq \{x^{i \bmod I}(p, \pi/n)\}_{i=1}^{nI}. \quad (27)$$

n-replicated profits equal

$$\pi^{(n)}(p, x(\cdot)) \triangleq \pi : p \cdot x(p, \pi) = np \cdot s(p) \quad (28)$$

$$\begin{aligned} &= n(\pi : p \cdot x(p, \pi)/n = p \cdot s(p)) \\ &= n\pi(p, x(\cdot)/n). \end{aligned} \quad (29)$$

Observe that individual and aggregate demand functions, and demand function profiles, are admissible in \mathcal{E} iff their n -replicated equivalents are admissible in $\mathcal{E}^{(n)}$.

In this context, with individuals partially internalizing the impacts of their purchases on total production levels, the relevant notion of market equilibrium is not Walrasian equilibrium, nor even the “Walrasian equilibrium with slack” (WES) used to deal with nonsatiation. The alternative equilibrium concept, however, will build closely on WES. Let us now therefore define WES more precisely.

Definition 4. *Given aggregate demand and supply functions $x(p, \pi)$ and $s(p)$, implicit demand equals*

$$\chi(p) \triangleq x(p, \pi(p, x(\cdot))), \quad (30)$$

and excess demand equals

$$z(p) \triangleq \chi(p) - s(p). \quad (31)$$

Let

$$\mathcal{I} \triangleq [I_{L-1}, 0_{L-1}] : \quad (32)$$

the $(L-1) \times L$ matrix whose first $L-1$ columns constitute the $(L-1)$ -dimensional identity matrix and whose last column is the zero vector.

Definition 5. *Given price vector p , the corresponding normalized price vector equals*

$$\hat{p} \triangleq \mathcal{I}p/p_L. \quad (33)$$

Given normalized price vector \hat{p} ,

$$\dot{\hat{p}} \triangleq (\hat{p}, 1) \quad (= p/p_L, \text{ if } p \text{ is given}), \quad (34)$$

and normalized excess demand equals

$$\hat{z}(\hat{p}) \triangleq \mathcal{I}z(\dot{\hat{p}}). \quad (35)$$

Definition 6. Price vector \bar{p} is a Walrasian equilibrium with slack (WES) of economy \mathcal{E} and aggregate demand function $x(\cdot)$ if $z(\bar{p}) = 0$.

Normalized price vector \hat{p} is a normalized WES of $(\mathcal{E}, x(\cdot))$ if $\hat{z}(\hat{p}) = 0$.

Definition 7. A WES \bar{p} of economy \mathcal{E} and aggregate demand function $x(\cdot) \in \mathbb{A}(\mathcal{E})$ is regular if the Jacobian of $\hat{z}(\cdot)$ exists and is nonsingular at \hat{p} .

Given an economy \mathcal{E} , an aggregate demand function $\bar{x}(\cdot) \in \mathbb{A}(\mathcal{E})$, and a WES \bar{p} of $(\mathcal{E}, \bar{x}(\cdot))$, let us also introduce the following shorthand:

$$\bar{\pi} \triangleq \pi(\bar{p}, \bar{x}(\cdot)). \quad (36)$$

3 Equilibrium characterization

Proposition 1 (Existence of WES).

There exists a WES $\bar{p} \gg 0$ of any economy \mathcal{E} and aggregate demand function $x(\cdot) \in \mathbb{A}(\mathcal{E})$. Furthermore, \bar{p} is a WES of $(\mathcal{E}, x(\cdot))$ iff \bar{p} is a WES of $(\mathcal{E}^{(n)}, x^{(n)}(\cdot)) \forall n \geq 1$, and iff \hat{p} is a normalized WES of $(\mathcal{E}, x(\cdot))$.

Proof. See Appendix A.1. □

Given an economy

$$\mathcal{E} = ((\{\omega^i\}_{i=1}^{I-1}, \bar{\omega}^I), \{u^i(x^i, s)\}_{i=1}^I, \{\theta^i\}_{i=1}^I, y(p)), \quad (37)$$

let

$$\mathcal{E}(\omega^I) \triangleq (\{\omega^i\}_{i=1}^I, \{u^i(x^i, s)\}_{i=1}^I, \{\theta^i\}_{i=1}^I, y(p)). \quad (38)$$

Given $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E})$, for some $i < I$, observe that $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E}(\omega^I))$. Let $\bar{x}^I(p, \pi, \omega^I)$ denote the unique admissible demand function for I in $\mathcal{E}(\omega^I)$.

Proposition 2 (Almost always all interior WES are regular, given \mathcal{C}^1 demands).

Given economy \mathcal{E} , consider $\{x^i(\cdot)\} \in \{\mathbb{A}^i(\mathcal{E})\}$ for which each $x^i(\cdot)$ is \mathcal{C}^1 . Then for almost every $\omega^I \gg 0$, every WES \bar{p} of $(\mathcal{E}(\omega^I), x^{-I}(\cdot) + \bar{x}^I(\cdot, \omega^I))$ for which $s(\bar{p}) \gg 0$ is regular. Furthermore, \bar{p} is a regular WES of $(\mathcal{E}, x(\cdot))$ iff \bar{p} is a regular WES of $(\mathcal{E}^{(n)}, x^{(n)}(\cdot)) \forall n \geq 1$.

Proof. See Appendix A.2. □

The assumption that $x^i(\cdot)$ is \mathcal{C}^1 for $i : w^i(\cdot) \neq 0$ can simply be imposed. The assumption that $x^i(\cdot)$ is \mathcal{C}^1 when $w^i(\cdot) = 0$ implies a restriction on $u^i(\cdot)$, for such i , beyond those we have made so far. See Kreps (2012, pp. 274–277) for (local) conditions on $u^i(\cdot)$ for which the price-/profit-taking $x^i(\cdot)$ is (locally) \mathcal{C}^1 .

Proposition 3 (Local regular WES under demand perturbations).

Let \bar{p} be a regular WES of economy \mathcal{E} and locally \mathcal{C}^1 demand function profile $\{\bar{x}^i(\cdot)\} \in \{\mathbb{A}(\mathcal{E})\}$. Then there exists an $\epsilon > 0$ and an $\underline{n} \geq 1$ such that

- for any i and $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E})$ that is \mathcal{C}^1 around $(\bar{p}, \bar{\pi})$, for $n \geq \underline{n}$ there exists a regular normalized WES \hat{p} of $(\mathcal{E}^{(n)}, \bar{x}^{-i(n)}(\cdot) + x^{i(n)}(\cdot))$ within the ϵ -neighborhood of \hat{p} ; and
- within the ϵ -neighborhood of \hat{p} , \hat{p} is the unique normalized WES.

Proof. See Appendix A.3. □

Definition 8. An economy \mathcal{E} is large with respect to $\{\bar{x}^i(\cdot)\} \in \{\mathbb{A}^i(\mathcal{E})\}$ and WES \bar{p} of $(\mathcal{E}, \bar{x}(\cdot))$ if Proposition 3 holds for $(\mathcal{E}, \bar{x}(\cdot), \bar{p})$ given $\underline{n} = 1$.

Given economy \mathcal{E} , demand functions $\{\bar{x}^i(\cdot)\} \in \{\mathbb{A}^i(\mathcal{E})\}$, and regular WES \bar{p} such that each $\bar{x}^i(\cdot)$ is locally \mathcal{C}^1 , choose n such that $\mathcal{E}^{(n)}$ is large with respect to $\{\bar{x}^i(\cdot)\}^{(n)}, \bar{p}$. Denote the locally unique normalized WES of $(\mathcal{E}^{(n)}, \bar{x}^{-i(n)}(\cdot) + x^{i(n)}(\cdot))$, for some locally \mathcal{C}^1 $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E})$, by $\hat{p}_{\bar{p}}^{(n)}(x^i(\cdot))$. Let

$$p_{\bar{p}}^{(n)}(x^i(\cdot)) \triangleq \bar{p}_L(\hat{p}_{\bar{p}}^{(n)}(x^i(\cdot)), 1). \quad (39)$$

Given locally \mathcal{C}^1 demand function $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E})$, denote the utility i achieves by demand function $x^{i(n)}$ in the replicated economy by

$$u_{\bar{p}}^{i(n)}(x^i(\cdot)) \triangleq u^i\left(x^{i(n)}\left(p_{\bar{p}}^{(n)}(x^i(\cdot)), \pi^{(n)}(p_{\bar{p}}^{(n)}(x^i(\cdot)), \bar{x}^{-i(n)}(\cdot) + x^{i(n)}(\cdot))\right), ns\left(p_{\bar{p}}^{(n)}(x^i(\cdot))\right)\right). \quad (40)$$

We can now define a competitive equilibrium concept suited to the case in which consumers care about their impacts on supply.

Definition 9. Let \bar{p} be a regular WES of economy \mathcal{E} and locally \mathcal{C}^1 demand function profile $\{\bar{x}^i(\cdot)\} \in \{\mathbb{A}(\mathcal{E})\}$. Then $(\bar{p}, \{\bar{x}^i(\cdot)\})$ is a competitive equilibrium with supply externalities (CESE) of \mathcal{E} if, for all i and all locally \mathcal{C}^1 $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E})$,

$$\lim_{n \rightarrow \infty} \left[u_{\bar{p}}^{i(n)}(x^i(\cdot)) - u_{\bar{p}}^{i(n)}(\bar{x}^i(\cdot)) \right] \leq 0. \quad (41)$$

That is, $(\bar{p}, \{\bar{x}^i(\cdot)\})$ is a CESE if, in response to $\bar{x}^{-i}(\cdot)$, each i chooses an admissible demand function $x^i(\cdot)$ that would be optimal for i if the economy were “very large” relative to i . In effect, $x^i(\cdot)$ is the demand function i chooses when he acts as a price-taker (because his impacts on prices are infinitesimal) but not as a quantity-taker (because his impacts on prices affect the behavior of infinitely many other agents, resulting in positive quantity effects even in the limit).

Let $\mathbf{0}_L$ denote the L -length vector of zeroes; let $D(a)$, for an arbitrary vector a , denote the diagonal $\text{Len}(a) \times \text{Len}(a)$ matrix M with $M_{\ell\ell} = a_\ell$ for $\ell = 1, \dots, \text{Len}(a)$;

and let $J_f(X)$ denote the Jacobian of function f at X . Then, given an economy \mathcal{E} , define the following terms:

$\delta(p, x(\cdot)) \triangleq$ The gradient of the aggregate Engel curve at $(p, \pi(p, x(\cdot)))$, i.e.
 $\nabla_{\pi} x(p, \pi(p, x(\cdot)))$

$G(p, x(\cdot)) \triangleq$ The generalized inverse G of $-J_z(p)$, where $z(\cdot)$ is implied (42)
 by $(\mathcal{E}, x(\cdot))$, with $G\delta(p, x(\cdot)) = \mathbb{0}_L$ and whose bottom row
 equals $\mathbb{0}_L^T$

or

$\sigma(p) \triangleq$ The matrix of cross-price elasticities of supply at p
 $\varepsilon(p, x(\cdot)) \triangleq$ The matrix of cross-price elasticities of implicit demand at p
 $\phi(p, x(\cdot)) \triangleq$ The generalized inverse ϕ of $\sigma(p) - \varepsilon(p, x(\cdot))$ with
 $\phi D(s(p))^{-1} \delta(p, x(\cdot)) = \mathbb{0}_L$ and whose bottom row equals $\mathbb{0}_L^T$

$$\psi^i(p, x(\cdot)) \triangleq (J_s(p) G(p, x(\cdot)))^T \nabla w^i(s(p)) \quad \text{or, equivalently,} \quad (43)$$

$$= (\sigma(p) \phi(p, x(\cdot)))^T \nabla w^i(s(p))$$

$\psi(p, x(\cdot)) \triangleq$ The $L \times I$ matrix with column i equal to $\psi^i(p, x(\cdot))$

We can now characterize CESEs relatively simply.

Proposition 4 (Characterization of CESE).

Let \bar{p} be a regular WES of economy \mathcal{E} and locally \mathcal{C}^1 demand function profile $\{\bar{x}^i(\cdot)\} \in \{\mathbb{A}(\mathcal{E})\}$. Then $(\bar{p}, \{\bar{x}^i(\cdot)\})$ is a CESE of \mathcal{E} iff

$$\bar{x}^i(\bar{p}, \bar{\pi}) = \operatorname{argmax}_{x^i} \left(v^i(x^i) + \psi^i(\bar{p}, \bar{x}(\cdot)) \cdot x^i \right) \quad \Bigg| \quad \bar{p} \cdot x^i \leq \bar{p} \cdot \omega^i + \theta^i \bar{\pi} \quad \forall i. \quad (44)$$

Proof. See Appendix A.4. □

An intuition for the result is as follows.

Starting from a WES \bar{p} , suppose i adjusts his demand around \bar{p} by a just-feasible Δx^i : i.e., by a value Δx^i satisfying $\bar{p} \cdot \Delta x^i = 0$. The impact of this demand-adjustment on prices will, when i is a negligible part of the economy, be linear in Δx^i , and can therefore be represented by $\Delta p = G \Delta x^i$ for some matrix G . G must satisfy

$$-J_z G \Delta x^i = \Delta x^i \quad \forall \Delta x^i : \bar{p} \cdot \Delta x^i = 0. \quad (45)$$

That is, the equilibrium price impact of demand-shift Δx^i must motivate shifts in supply ($J_s \Delta p$) and in others' demands ($J_\chi \Delta p$) such that the gap between total supply and others' demands changes by precisely Δx^i . Note that $-J_z = J_s - J_\chi$.

G must therefore be a generalized inverse of $-J_z$. Furthermore, the equilibrium price impact of a demand-shift along the gradient of the aggregate Engel curve must be zero: such a shift can be precisely accommodated by changing the aggregate profit rate, and thus others' demands, without affecting prices or supply. We thus have $G\delta = 0_L$. Finally, to restrict the space of prices under consideration to normalized price vectors whose L^{th} entries equal 1 (or, more precisely, to restrict the space of price-changes to those in which $\Delta p_L = 0$, so that p_L remains fixed at any given value), the bottom row of G must equal zero. There is a unique generalized inverse of $J_s - J_x$ satisfying these two conditions. The conditions thus identify G .

Then, i 's marginal "ethical impact" of demand-shift Δx^i , starting from a given WES, equals $\psi^i \cdot \Delta x^i$, where

$$\psi^{iT} = \nabla w^i \cdot J_s G : \quad (46)$$

G converts the demand-shift to a price-shift, J_s converts the price-shift to a supply-shift, and ∇w^i converts the supply-shift to a marginal ethical impact from i 's perspective. Transposing (46) yields (43).

3.1 Example

To illustrate the equilibrium concept, consider the economy $\mathcal{E}^{(n)}$, an arbitrary n -replication of the following economy \mathcal{E} with $I = 2$ individuals and $L = 2$ goods:

$$\omega^i = (3/2, 0) \quad \forall i, \quad (47)$$

$$v^1(x^1) = \ln(x_1^1) + x_1^1/3 + x_2^1, \quad (48)$$

$$v^2(x^2) = \ln(x_1^2) + x_2^2; \quad (49)$$

$$w^1(s) = s_1 + 2s_2, \quad (50)$$

$$w^2(s) = 0; \quad (51)$$

$$\theta^i = 1/2 \quad \forall i \quad (52)$$

and a production possibility frontier given by

$$y_2 = 4 - \frac{4}{1 - y_1}, \quad y_1 \geq -3, \quad y_2 \geq 0, \quad (53)$$

so that profit-maximizing production equals

$$y(p) = \left(1 - 2\sqrt{\frac{p_2}{p_1}}, 4 - 2\sqrt{\frac{p_1}{p_2}} \right) \quad (54)$$

as long as $p_1/p_2 \in [1/4, 4]$ to ensure an interior solution. (Recall from the discussion following (15) that profit-maximizing production maximizes $p \cdot y$, even when profits do not equal $p \cdot y$.)

By definition, supply is therefore

$$s(p) = \left(4 - 2\sqrt{\frac{p_2}{p_1}}, 4 - 2\sqrt{\frac{p_1}{p_2}} \right), \quad (55)$$

as long as $p_1/p_2 \in [1/4, 4]$.

Let $\bar{p} \triangleq (1, 1)$, $\bar{\psi}^1 \triangleq (-1/3, 0)$, and $\bar{\psi}^2 \triangleq (0, 0)$, and let

$$\bar{x}^i(p, \pi) \triangleq \operatorname{argmax}_{x^i} \tilde{u}^i(x^i) \mid p \cdot x^i \leq p \cdot \omega^i + \theta^i \pi \quad \forall i \quad (56)$$

$$\text{where } \tilde{u}^i(x^i) \triangleq v^i(x^i) + \bar{\psi}^i \cdot x^i = \ln(x_1^i) + x_2^i \quad \forall i \quad (57)$$

$$\implies \bar{x}_1^i(p, \pi) = \min \left(\frac{3p_1 + \pi}{2p_1}, \frac{p_2}{p_1} \right) \quad \forall i, \quad (58)$$

$$\bar{x}_2^i(p, \pi) = \max \left(0, \frac{3p_1 + \pi}{2p_2} - 1 \right) \quad \forall i. \quad (59)$$

Note that the specified externality weight of $\bar{\psi}_1^1 = -1/3$ cancels out the $+x_1^1/3$ term of (48), leaving the maximand $\tilde{u}^i(x^i)$ equal for both i . The individuals' demand functions are then identical because the individuals also have the same endowments and profit shares.

We will now show that $(\bar{p}, \{\bar{x}^i(\cdot)\}^{(n)})$ is a CESE of $\mathcal{E}^{(n)}$.

To take care of the preliminaries, first observe that demand function profile $\{\bar{x}^i(\cdot)\}$ is admissible in \mathcal{E} . As noted following Definition 3, it follows that $\{\bar{x}^i(\cdot)\}^{(n)}$ is admissible in $\mathcal{E}^{(n)}$. Observe also that $\{\bar{x}^i(\cdot)\}$, and thus $\{\bar{x}^i(\cdot)\}^{(n)}$, is locally \mathcal{C}^1 around (\bar{p}, π) for any $\pi \geq 0$.

Next, let us confirm that \bar{p} is a WES of \mathcal{E} and demand function profile $\{\bar{x}^i(\cdot)\}$. From (54), $y(\bar{p}) = (-1, 2)$. Also, $\tilde{u}^i(x^i)$ is strictly increasing for both i , so both individuals exhaust their budgets at any p, π . Thus $\bar{\pi} \triangleq \pi(\bar{p}, \bar{x}(\cdot)) = \bar{p} \cdot y(\bar{p}) = 1$. $\bar{x}(\bar{p}, \bar{\pi}) = s(\bar{p}) = (2, 2)$, confirming that \bar{p} is a WES.

Then, by substituting $p \cdot y(p)$ for π in (59) and summing across i , we have aggregate implicit demand $\bar{\chi}(p)$. From here, though we will not work through the details, it is straightforward to find excess demand $\bar{z}(\cdot)$ ($\triangleq \bar{\chi}(\cdot) - s(\cdot)$). With normalized excess demand $\hat{z}(\cdot)$ defined as in Definition 5, we can then confirm that \bar{p} is regular by Definition 7.

Finally, by Proposition 2, it follows that \bar{p} is also a regular WES of $(\mathcal{E}^{(n)}, \{\bar{x}^i(\cdot)\}^{(n)})$.

What makes the proposed $(\bar{p}, \{\bar{x}^i(\cdot)\})$ a CESE and not merely a regular WES is that $\bar{\psi} = \psi(\bar{p}, \bar{x}(\cdot))$. This can be calculated from the Definition (43) of $\psi(\cdot)$ (and the prior Definition 42 of $G(\cdot)$), but a more intuitive approach is as follows.

Let $\tilde{G}^{(n)}(p, x(\cdot))$ denote an $L \times L$ (here, 2×2) matrix mapping individual demand-changes into changes to equilibrium prices, in economy $\mathcal{E}^{(n)}$, given initial conditions

in which aggregate demand is $x(\cdot)$ and prices are a regular WES p of $(\mathcal{E}^{(n)}, x^{(n)}(\cdot))$.⁵ Since excess demand is h.o.d. 0, we can without loss of generality fix p_L , i.e. require that $\tilde{G}(\cdot)$ have a bottom row of zeroes.

Now observe that $\bar{x}_2^i(\bar{p}, \bar{\pi}) > 0 \forall i$. The quasilinearity of (57) guarantees that, at $(\bar{p}, \bar{\pi})$, each i 's marginal purchases are exclusively of good 2. Therefore

$$\delta(\bar{p}, \bar{x}(\cdot)) = (0, 1/\bar{p}_2)^T = (0, 1)^T. \quad (60)$$

This in turn implies that if one individual i reduces x_2^i , this simply increases profits, which others spend entirely on good 2. Supply levels do not change, and neither do prices. (A price-change would induce a supply-change.) In other words, $G(\bar{p}, \bar{x}(\cdot))(0, -1)^T = (0, 0)^T$. The upper-right entry of $G(\bar{p}, \bar{x}(\cdot))$ therefore equals zero.

The only remaining unknown entry of $G(\bar{p}, \bar{x}(\cdot))$ is its upper-left, representing the extent to which marginal purchases of good 1 increase the price of good 1. To find it, we will consider a marginal individual demand-change proportional to $(1, -1)^T$. Since $\bar{p}_1 = \bar{p}_2$, this demand-change by i is orthogonal to \bar{p} .

By $s(p)$ from (55), $\partial s_1(p)/\partial p_1 = 1$ and $\partial s_2(p)/\partial p_1 = -1$. That is, recalling that supply in $\mathcal{E}^{(n)}$ equals $ns(p)$, each marginal unit increase in p_1 from the \bar{p} baseline induces an n -unit increase in the supply of good 1 and an n -unit decrease in the supply of good 2 in economy $\mathcal{E}^{(n)}$.

Likewise, by (58)–(59), each marginal unit increase in p_1 induces a 1-unit decrease in demand for good 1 by all $2n - 1$ consumers other than i . (Holding profits fixed, it also induces a 1-unit increase in demand for good 2. We have not shown that profits will in fact remain fixed as prices change; but we know that profit-changes will not affect demand for good 1, as long as all consumers are consuming a positive quantity of good 2.)

To maintain market clearing, p_1 must rise by just enough to induce an increase in s_1 , and a decrease in x_1^{-i} , which sums to 1: the additional unit of good 1 which consumer i has resolved to buy. That is, we must have

$$n\Delta p_1 + (2n - 1)(\Delta p_1) = 1 \quad (61)$$

$$\implies \Delta p_1 = \frac{1}{3n - 1}. \quad (62)$$

As noted above, changes in demand for good 2, starting from the baseline of $(\bar{p}, \bar{x}(\cdot))$, have no impact on supply levels. It follows that $\psi_2^1(\bar{p}, \bar{x}(\cdot)) = \psi_2^2(\bar{p}, \bar{x}(\cdot)) = 0$, as

⁵The tilde distinguishes $\tilde{G}^{(n)}(\cdot)$ from $G(\cdot)$, which technically maps individual demand-changes into what equilibrium price-changes *would* be if the individual specifying a demand-change also responded to the price-change she herself induces. This distinction vanishes for large n .

desired. Moreover, since $w^2(s) = 0$, $\psi_1^2(\bar{p}, \bar{x}(\cdot)) = 0$ by (43), as desired. All that remains is to show that $\psi_1^1(\bar{p}, \bar{x}(\cdot)) = -1/3$.

Recall that the marginal individual demand-shift of $(1, -1)$ induces a marginal price-shift of $\Delta p_1 = \frac{1}{3n-1}$. This price-shift, in turn, induces a marginal $\frac{n}{3n-1}$ -unit increase in the equilibrium supply of good 1 and $\frac{n}{3n-1}$ -unit decrease in the equilibrium supply of good 2. Recalling that $w^1(s) = s_1 + 2s_2$, the marginal ethical impact of marginal individual demand-shift $(1, -1)$, from the perspective of consumer 1 (or any of her clones), equals $-\frac{n}{3n-1}$. As $n \rightarrow \infty$, this ethical impact approaches $-1/3$. By definition, therefore, $\psi_1^1(\bar{p}, \bar{x}(\cdot)) - \psi_2^1(\bar{p}, \bar{x}(\cdot)) = -1/3$. But $\psi_2^1(\bar{p}, \bar{x}(\cdot)) = 0$. Thus $\psi_1^1(\bar{p}, \bar{x}(\cdot)) = -1/3$, as desired.

With $\bar{\psi} = \psi(\bar{p}, \bar{x}(\cdot))$, $(\bar{p}, \bar{x}(\cdot))$ is a CESE of \mathcal{E} and any of its replications.^{6,7}

As this example illustrates, the impacts of consumer behavior after accounting for general equilibrium effects can differ substantially from the impacts one finds when one entirely ignores substitution by other parties. Inspecting $w^1(s)$ alone, one might expect that consumer 1 assigns an externality weight of 1 to purchasing a unit of good 1, and an externality weight of 2 to purchasing a unit of good 2, on any margin. Here, by contrast, we find that she assigns a negative weight to good 1 and a weight of 0 to good 2.

Likewise, external impacts in general equilibrium can differ substantially from those found after accounting only for partial equilibrium effects. (55) and (58) record an upward-sloping supply curve and a downward-sloping demand curve for both goods, respectively, around $p = (1, 1)$. A partial equilibrium analysis would therefore conclude that, from consumer 1's perspective, the ethical impact of purchasing a unit of good 1 given prices $(1, 1)$ lay somewhere in $(0, 1)$, and the ethical impact of purchasing a unit of good 2 lay in $(0, 2)$.

3.2 Discussion

Proposition 4 states that each consumer i is indifferent between all demand functions which demand, at the equilibrium price and profit level, the basket x^i that maximizes $v^i(x^i) + \psi^i(\bar{p}, \bar{x}(\cdot)) \cdot x^i$ subject to her budget constraint. The proposition thus offers little guidance as to what sorts of behavior we might expect to see in equilibrium. The (cross-)price elasticities of demand i chooses *around* the equilibrium price and profit level are of no consequence for i , but because they affect $x(\cdot)$ and thus ψ^j for $j \neq i$, they affect other consumers' best-response demand functions. That is, in a CESE, consumers' demands are highly sensitive to their fellow consumers' arbitrary choices of threatened out-of-equilibrium behavior. Exotic behavior may therefore be

⁶ $(\bar{p}, \bar{\psi})$ is also an RCESE of \mathcal{E} and its replications, where RCESE is a refinement of CESE defined in §4 below. Furthermore, though we have not shown this here, this RCESE is unique.

⁷Detailed derivations and instructive graphs regarding this example can be found [here](#). The source code, which can be used to explore similar examples, can be found [here](#).

motivated in equilibrium by mutually best-responding threats with no basis in anyone's preferences over supply or own consumption. A refinement of CESE, designed to rule out these “non-credible threats”, will be discussed in §4.

It is worth noting that even in the absence of such a refinement, Proposition 4 offers complete guidance as to what *basket* an individual in a large economy, with given preferences and a given endowment, should buy at the equilibrium prices and profit level. Again, the individual should buy the x^i that maximizes $v^i(x^i) + \psi^i(\bar{p}, \bar{x}(\cdot)) \cdot x^i$ subject to her budget constraint. If our goal is simply to offer advice to informed consumers with given ethical preferences (including ourselves), therefore, our analysis can end here.

A second weakness of the above result, however, is that the informational requirements for computing ψ^i are demanding. In particular, i must know the gradient of the aggregate Engel curve and either (a) the Jacobians of supply and demand with respect to price or, equivalently, (b) the cross-price elasticity matrices of supply and demand and the aggregate supply levels. Thankfully, these informational requirements can under some circumstances be relaxed. Circumstances under which an ethical consumer need not know the gradient of the aggregate Engel curve will be discussed following Proposition 6. Circumstances under which she need not know the complete Jacobians or cross-price elasticity matrices would be a valuable avenue for further research.

4 Equilibrium refinement

Let us refer to ψ^i as i 's externality vector. Let ψ denote the $L \times I$ externality matrix whose column i equals ψ^i .

Given ψ^i , let us refer to

$$\tilde{u}^i(x^i, \psi^i) \triangleq v^i(x^i) + \psi^i \cdot x^i \quad (63)$$

as i 's quasi-utility function. $\tilde{u}^i(\cdot)$, for any $\psi^i \in \mathbb{R}^L$, represents preferences over $x^i \in \mathbb{R}_{\geq 0}^L$. More precisely, it represents all-things-considered preferences over purchasing choices: preferences incorporating both the private benefits of consuming a given bundle and the external impacts of the production-changes induced by purchasing that bundle.

As discussed in §3.2, we would like a refinement of CESE in which individuals choose demand functions that maximize (63)—where ψ^i is defined as in (43)—not only at $(\bar{p}, \bar{\pi})$ but at all (p, π) , or at least all (p, π) near $(\bar{p}, \bar{\pi})$. We can motivate such demand functions by giving each consumer a small degree of uncertainty about the prices and profits she will face, while ensuring that the external impacts of a given consumption decision are independent of these prices and profits.

Note the importance of maintaining this independence. Without it, an unexpected (p, π) may reveal a condition in which, as a result of shocks to production or

others' preferences, the ethical impact of purchasing a given bundle differs from what it would be at the equilibrium $(\bar{p}, \bar{\pi})$. Thus an individual i with ethical preferences may wish, conditional on some unexpected (p, π) , to demand a bundle other than the one that maximizes (63) for a ψ^i fixed at its definition from (43). In other words, in a world with shocks that affect the implications of one's purchasing decisions, i would want to make her demands a function not only of prices and profits but also of the shocks; and if we forbid this, i would want to make her demands sensitive to the shocks to the extent that they can be discovered from realized prices and profits. But .

INFORMAL NOTES

Here I want to define an refinement of CESE in which individuals choose demand functions that maximize (63)—where ψ^i is defined as in (43)—not only at $(\bar{p}, \bar{\pi})$ but at (preferably) all (p, π) , or at least all (p, π) locally around $(\bar{p}, \bar{\pi})$. I expect that the definition of and justification for this refinement will look something like the following:

Definition 10. A *robust CESE* (RCESE) is a CESE in which each individual i chooses a demand function i that satisfies (41), but in expected utility terms, given a bit of uncertainty about what (p, π) she will face.

Proposition 5 (Refinement to RCESE).

In any RCESE, individuals choose the $x^i(\cdot)$ that maximizes (63) not only at $(\bar{p}, \bar{\pi})$ but at [all (p, π) , or all (p, π) locally around $(\bar{p}, \bar{\pi})$].

As long as people's demand functions maximize (63) around equilibrium prices and profits, the Jacobians of their demand functions in equilibrium are pinned down, not just the quantities they demand precisely at equilibrium. Thus J_x is pinned down. So without loss of generality, we can just consider cases in which each i actually chooses the $x^i(\cdot)$ that maximizes (63) everywhere. A commitment to choosing different demands far away from equilibrium will have no bearing on others' best-response demand functions.

Even if a formal refinement strategy like the above doesn't work, though, we can always just directly define an RCESE as one in which each i chooses the $x^i(\cdot)$ that maximizes (63) everywhere. (Maybe in this case it should just be called a "linear CESE", or LCESE, since we won't have shown any formal sense in which they're more robust than other CESEs.) The value of Proposition 4 will then essentially just be in demonstrating that (L/R)CESEs are actually CESEs.

END OF INFORMAL NOTES

Note that, if $w^i(\cdot) = 0$, $\psi^i(\cdot) = 0$ as well. In any RCESE, therefore, individuals without ethical preferences adopt the demand functions that maximize $v^i(\cdot)$, as individuals are assumed to do in the conventional Walrasian setting. Unlike in the setting of §3 in which individuals choose arbitrary admissible $x^i(\cdot)$, we do not have to assume this explicitly.

RCESE is, therefore, a relatively straightforward generalization of Walrasian equilibrium—given certain technicalities (namely regularity and locally \mathcal{C}^1 demand functions)—to the case in which individuals have preferences over total supply levels. This will also be stated formally in Proposition 7, in the section below.

In any event, going forward we will restrict our attention to RCESEs. This lets us reframe much of the model of §2 in simpler terms.

In particular, instead of positing that each i chooses an arbitrary admissible demand function $x^i(\cdot)$, we can simply posit that i chooses a vector ψ^i . We can then define the demand function compatible with a given ψ^i as

$$x_{[\psi^i]}^i(p, \pi) \triangleq \operatorname{argmax}_{x^i} \tilde{u}^i(x^i, \psi^i) \mid p \cdot x^i \leq p \cdot \omega^i + \theta^i \pi,^8 \quad (64)$$

and let $x_{[\psi]}(\cdot)$ denote the sum of individual $x_{[\psi^i]}^i(\cdot)$. With

$$\pi(p, \psi) \triangleq \pi(p, x_{[\psi]}(\cdot)), \quad (65)$$

we can then define the implicit demand function compatible with a given ψ as

$$\chi_{[\psi]}^i(p) \triangleq \operatorname{argmax}_{x^i} \tilde{u}^i(x^i, \psi^i) \mid p \cdot x^i \leq p \cdot \omega^i + \theta^i \pi(p, \psi), \quad (66)$$

let $\chi_{[\psi]}(\cdot)$ denote the sum of individual $\chi_{[\psi^i]}^i(\cdot)$, and let

$$z_{[\psi]}(p) \triangleq \chi_{[\psi]}(p) - s(p). \quad (67)$$

Finally, as with $\pi(\cdot)$ in (65), we can define

$$\delta(p, \psi) \triangleq \delta(p, x_{[\psi]}(\cdot)), \quad (68)$$

and analogously extend (43) and the preceding terms—defined there as functions of admissible aggregate demand functions $x(\cdot)$ —to be defined with externality matrices, and not only explicit aggregate demand functions, in their second arguments.

Finally, we can define (p, ψ) to be an RCESE if $(p, \{x_{[\psi^i]}^i(\cdot)\})$ is an RCESE. Or, equivalently,

⁸Expression (64) will be defined for all ψ^i if $v^i(\cdot)$ is strictly concave. If $v^i(\cdot)$ is quasilinear in good L , $x_{[\psi^i]}^i(\cdot)$ will be defined as long as $\psi_L^i \neq -\partial v^i / \partial x_L^i$. Observe that $x_{[\psi^i]}^i(\cdot)$ will be admissible whenever it is defined.

Definition 11. Given $\bar{\psi}$, let \bar{p} be a regular WES of economy \mathcal{E} and demand function profile $\{x_{[\bar{\psi}^i]}^i(\cdot)\}$. Then $(\bar{p}, \bar{\psi})$ is an RCESE of \mathcal{E} if each $x_{[\bar{\psi}^i]}^i(\cdot)$ is defined and locally \mathcal{C}^1 around $(\bar{p}, \bar{\pi})$, with

$$\psi(\bar{p}, \bar{\psi}) = \bar{\psi}. \quad (69)$$

As noted at the end of §3, there are circumstances under which a consumer does not need to know the gradient of the aggregate Engel curve in order to compute her optimal demands. One such circumstance is as follows.

Proposition 6 (Non-satiation in RCESE given no aggregate inferior goods). *Given an RCESE $(\bar{p}, \bar{\psi})$, suppose $\delta(\bar{p}, \bar{\psi}) \geq \mathbb{0}_L$. Then $\bar{\psi}^i \not\leq 0 \forall i$, and all individuals exhaust their budgets.*

Proof. For all (p, ψ) , we have $G(p, \psi)\delta(p, \psi) = \mathbb{0}_L$, and thus

$$\psi^i(p, \psi) \cdot \delta(p, \psi) = \nabla w^i(s(p)) \cdot J_s(p)G(p, \psi)\delta(p, \psi) = 0 \quad \forall i. \quad (70)$$

Since $\delta(\bar{p}, \bar{\psi}) \geq \mathbb{0}_L$, it follows that $\bar{\psi}^i(\bar{p}, \bar{\psi}) \not\leq 0$. Because $v^i(\cdot)$ is strictly increasing in all goods for all i , it follows that for every i , there is at least one good in which i is nonsatiated in equilibrium. Therefore all individuals exhaust their budgets. \square

Proposition 6 tells us that as long as $\delta \geq 0$ —that is, as long as each good ℓ is not “inferior in aggregate”, in that x_ℓ is not locally decreasing in the profit rate—then each i can infer that it will be optimal for her to exhaust her budget. As a result, i does not need to choose her demands by constructing G , and then ψ^i , using known δ . Instead, i can recognize that *any* generalized inverse \tilde{G} of $-J_z$ satisfies $-J_z(\bar{p})\tilde{G}\Delta x^i = \Delta x^i$ for $\Delta x^i : \bar{p} \cdot \Delta x^i = 0$, and thus that \tilde{G} captures the price-impacts,⁹ and pins down the supply impacts, of demand choices among bundles that exhaust i ’s budget. Therefore, in this setting,

$$x_{[\psi^i]}(p, \pi) = \operatorname{argmax}_{x^i} \tilde{u}^i(x^i, \tilde{\psi}^i) \mid p \cdot x^i = p \cdot \omega^i + \theta^i \pi \quad (71)$$

for any $\tilde{\psi}^i = (J_s(\bar{p})\tilde{G})^T \nabla w^i(s(\bar{p}))$, where \tilde{G} is an arbitrary generalized inverse of $-J_z(\bar{p})$.

Corollary 6.1 (Non-satiation in RCESE given additive separability).

Suppose that, for all i , $v^i(\cdot)$ is additively separable (and that the standard conditions on $v^i(\cdot)$ are met). Then, in any RCESE $(\bar{p}, \bar{\psi})$, $\bar{\psi}^i \not\leq 0 \forall i$, and all individuals exhaust their budgets.

⁹Up to rescaling. Without loss of generality, we can restrict ourselves to price impacts with $\Delta p_L = 0$ by imposing that the bottom row of \tilde{G} equal $\mathbb{0}_L^T$.

Proof. The additive separability of the $v^i(\cdot)$ and thus of the $\tilde{u}^i(\cdot)$ implies that, at any (p, ψ) , no good is inferior for any i (given that $x_{[\bar{p}^i]}^i(\cdot)$ exists, which it must by the assumption that $(\bar{p}, \bar{\psi})$ is an RCESE). Therefore no good is inferior in aggregate. The result then follows from Proposition 6. \square

It may be tempting to try to guarantee that individuals exhaust their budgets by positing that $w^i(\cdot)$ is nondecreasing in at least one good for each i , or imposing some other assumption of this kind. Unfortunately, however, after accounting for the effects of buying one good on the equilibrium quantities of other goods, purchasing goods for which $w^i(\cdot)$ is nondecreasing (or even increasing) may do harm from i 's perspective, and purchasing goods for which $w^i(\cdot)$ is decreasing may do good. There does not appear to be any straightforward relationship between assumptions on the $\{w^i(\cdot)\}$ and satiation.

5 Equilibrium existence

As noted in §4, if there are no “ethical consumers”, then RCESE is essentially equivalent to Walrasian equilibrium, subject to certain technicalities. We can state this result more formally:

Proposition 7 (RCESE generalizes Walrasian equilibrium).

Let \bar{p} be a Walrasian equilibrium of an economy \mathcal{E} with $w^i(\cdot) = 0 \forall i$. Then, if \bar{p} is regular and $x_{[\bar{p}]}^i(p, \pi)$ is \mathcal{C}^1 around $(\bar{p}, \bar{p} \cdot y(\bar{p}))$ for all i , then $(\bar{p}, \mathbb{0}_{L \times I})$ is an RCESE of \mathcal{E} .

Proof. Because \bar{p} is a regular Walrasian equilibrium, \bar{p} is a regular WES (with $\pi = \bar{p} \cdot y(\bar{p})$). Because each $v^i(\cdot)$ is strictly quasiconcave, each $x_{[\bar{p}]}^i(\cdot)$ is defined; and each $x_{[\bar{p}]}^i(\cdot)$ is locally \mathcal{C}^1 by assumption. Finally, because each $w^i(\cdot) = 0$, each $\psi^i(\cdot) = \mathbb{0}_L$, by (43). Thus $\psi(\bar{p}, \mathbb{0}_{L \times I}) = \mathbb{0}_{L \times I}$. \square

Determining when an RCESE exists in a less trivial setting is difficult. However, we can show existence under certain conditions.

Recalling the definition of P in (4), let

$$\underline{p}_\ell \triangleq \min_{p \in P} p_\ell. \quad (72)$$

Because P is compact and $p \gg 0 \forall p \in P$, (72) is defined and positive for each ℓ .

Proposition 8 (Existence of RCESE under separability and quasilinearity).

Suppose every individual i has a consumption utility function $v^i(\cdot)$ of the form

$$v^i(x^i) = \sum_{\ell=1}^{L-1} \tilde{v}_\ell^i(x_\ell^i) + t^i \cdot x^i, \quad (73)$$

where, for all $\ell < L$, $\tilde{v}_\ell^i(\cdot)$ is \mathcal{C}^2 , strictly increasing, and strictly concave, and satisfies the lower and upper Inada conditions. Then, for each i , there exists a bound $\bar{\bar{\psi}}^i$ such that, if for all $\ell < L$

$$t_L^i > (t_\ell^i + \bar{\bar{\psi}}_\ell^i)/\underline{p}_\ell \text{ and} \quad (74)$$

$$\omega_\ell^i > \tilde{v}_\ell^{i'-1}(\underline{p}_\ell t_L^i - t_\ell^i - \bar{\bar{\psi}}_\ell^i), \quad (75)$$

then an RCESE $(\bar{p}, \bar{\psi})$ exists.

Furthermore, in any such RCESE, $\bar{\psi}_\ell^i \in [-\bar{\bar{\psi}}^i, \bar{\bar{\psi}}^i] \forall i, \ell < L$ and $\psi_L^i = 0 \forall i$.

Proof. See Appendix A.5. □

Appropriate $\bar{\bar{\psi}}^i$ bounds are constructed in the course of the proof, but unfortunately they cannot be stated simply.

Thus RCESEs exist under at least some conditions beyond the setting in which they reduce to Walrasian equilibria. Though the conditions required for Proposition 8 are highly restrictive, there does not appear to be any strong reason to believe that RCESEs do not exist much more widely.

6 References

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Appendices

A Proofs

A.1 Proof of Proposition 1

Given an economy \mathcal{E} and an admissible aggregate demand function $x(\cdot)$, consider the excess demand function

$$z(p) = x(p, \pi(p, x(\cdot))) - s(p)$$

as defined by (31).

First, let us show that $z(p)$ is continuous in p . This will follow directly from showing that $\pi(p, x(\cdot))$ is continuous in p .

Recall from (12) that

$$\pi(p, x(\cdot)) = \pi : Z(p, \pi, x(\cdot)) = 0, \quad (76)$$

where

$$Z(p, \pi, x(\cdot)) \triangleq p \cdot x(p, \pi) - p \cdot s(p). \quad (77)$$

Suppose by contradiction that $\pi(p, x(\cdot))$ is not continuous in p . Then there exists a \bar{p} and a $\delta > 0$ such that, for all $\epsilon > 0$,

$$\exists p \in N_\epsilon(\bar{p}) : \pi(p, x(\cdot)) \notin (\pi(\bar{p}, x(\cdot)) - \delta, \pi(\bar{p}, x(\cdot)) + \delta), \quad (78)$$

where $N_\epsilon(p)$ denotes the neighborhood of radius ϵ around p .

For every natural $n > 0$, we can choose a $p^n \in N_{1/n}(\bar{p})$ such that either

$$\pi(p^n, x(\cdot)) \geq \pi(\bar{p}, x(\cdot)) + \delta \text{ or} \quad (79)$$

$$\pi(p^n, x(\cdot)) \leq \pi(\bar{p}, x(\cdot)) - \delta. \quad (80)$$

There must thus be either an infinite subset ν of the naturals such that

$$\pi(p^n, x(\cdot)) \geq \pi(\bar{p}, x(\cdot)) + \delta \quad \forall n \in \nu \quad (81)$$

or an infinite subset ν of the naturals such that

$$\pi(p^n, x(\cdot)) \leq \pi(\bar{p}, x(\cdot)) - \delta \quad \forall n \in \nu. \quad (82)$$

Let ν be an infinite subset of the naturals such that (81) holds, and consider the sequence $\{(p^n, \pi(\bar{p}, x(\cdot)) + \delta)\}_{n \in \nu}$. Because $Z(p, \pi, x(\cdot))$ is strictly increasing in π , and $\pi : Z(p^n, \pi, x(\cdot)) = 0$ is no less than $\pi(\bar{p}, x(\cdot)) + \delta$, we have

$$Z(p^n, \pi(\bar{p}, x(\cdot)) + \delta, x(\cdot)) \leq 0 \quad \forall n \in \nu. \quad (83)$$

Because $Z(p, \pi, x(\cdot))$ is continuous in p , however, and because $p^n \rightarrow \bar{p}$,

$$\{Z(p^n, \pi(\bar{p}, x(\cdot)) + \delta, x(\cdot))\}_{n \in \nu} \rightarrow Z(\bar{p}, \pi(\bar{p}, x(\cdot)) + \delta) > 0, \quad (84)$$

with the inequality holding because $Z(\bar{p}, \pi(\bar{p}, x(\cdot)) + \delta) = 0$ and $Z(p, \pi, x(\cdot))$ is strictly increasing in π . (83) contradicts (84), so there is no infinite subset of the naturals such that (81) holds.

Analogous reasoning proves that there is no infinite subset of the naturals such that (82) holds.

Therefore $\pi(p, x(\cdot))$ is continuous in p . And thus $z(p)$, as the composition of continuous functions, is also continuous in p .

Because $x(p, \pi)$ is h.o.d. 0 in (p, π) and $s(p)$ is h.o.d. 0 in p , $Z(p, \pi, x(\cdot))$ is h.o.d. 0 in (p, π) . In particular, if $Z(p, \pi, x(\cdot)) = 0$, $Z(kp, k\pi, x(\cdot)) = 0 \forall k$. It follows that $\pi(p, x(\cdot))$ is h.o.d. 1 in p .

Thus

$$\begin{aligned} z(kp) &= x(kp, \pi(kp, x(\cdot))) - s(kp) \\ &= x(kp, k\pi(p, x(\cdot))) - s(kp) \\ &= x(p, \pi(p, x(\cdot))) - s(p) \\ &= z(p); \end{aligned} \tag{85}$$

$z(p)$ is h.o.d. 0 in p .

By construction of $\pi(\cdot)$, $p \cdot z(p) = 0 \forall p \gg 0$.

Because the production possibility set (and thus the supply possibility set) is compact,

$$\exists \bar{s} > 0 : s_\ell(p) < \bar{s} \quad \forall \ell, p \gg 0. \tag{86}$$

It then follows from the fact that $x_\ell(p, \pi) \geq 0 \forall \ell, p \gg 0, \pi \geq 0$ that

$$z_\ell(p) > -\bar{s} \quad \forall \ell, p \gg 0. \tag{87}$$

Consider a sequence of positive prices $\{p^n\} \rightarrow p$, for some $p \neq 0$ such that $p_\ell = 0$ for some ℓ . Suppose by contradiction that it doesn't hold that

$$\left\{ \max_k (x_k(p^n, \pi(p^n, x(\cdot)))) \right\}_n \rightarrow \infty.$$

Hence, there exists $M > 0$ such that $0 \leq x_k(p^n, \pi(p^n, x(\cdot))) \leq M$, for all n and all k . In particular, since individual demands are nonnegative, it follows that individual I 's demand for every good is also bounded from above by M , that is, $0 \leq x_k^I(p^n, \pi(p^n, x(\cdot))) \leq M$ for all n and all k .

Note that, since $\omega^I \gg 0$ and $p \neq 0$, it follows that $\lim_{n \rightarrow \infty} p^n \cdot \omega^i + \theta^i \pi(p^n, x(\cdot)) > 0$.

We now show that the upper bound M in the consumption of any good implies that I consumes zero of any of the goods whose price does not converge to zero. Let $\ell_0 \in L_0 = \{1 \leq \ell \leq L; p_\ell^n \rightarrow 0\}$ denote a good whose price converges to zero, and $\ell_1 \in \{1, \dots, L\} \setminus L_0$ denote a good whose price does not converge to zero. Note that if $\{x_{\ell_1}^I(p^n, \pi(p^n, x(\cdot)))\}_n$ is bounded from below by any positive value $\bar{x}_{\ell_1}^I > 0$, then,

since consumption for goods are bounded from above by M and u^I is \mathcal{C}^1 , strictly increasing, and concave, it follows that

$$\frac{\partial u^I}{\partial x_{\ell_1}}(x^I(p^n, \pi(p^n, x(\cdot)))) \leq \frac{\partial u^I}{\partial x_{\ell_1}}(M, \dots, M, \bar{x}_{\ell_1}^I, M, \dots, M) < \infty.$$

On the other hand, for good ℓ_0 , we have

$$\frac{\partial u^I}{\partial x_{\ell_0}}(x^I(p^n, \pi(p^n, x(\cdot)))) \geq \frac{\partial u^I}{\partial x_{\ell_0}}(Me_{\ell_0}) > 0.$$

Hence, we must have that, for n sufficiently large,

$$\frac{\frac{\partial u^I}{\partial x_{\ell_0}}(x^I(p^n, \pi(p^n, x(\cdot))))}{\frac{\partial u^I}{\partial x_{\ell_1}}(x^I(p^n, \pi(p^n, x(\cdot))))} \geq \frac{\frac{\partial u^I}{\partial x_{\ell_0}}(Me_{\ell_0})}{\frac{\partial u^I}{\partial x_{\ell_1}}(M, \dots, M, \bar{x}_{\ell_1}^I, M, \dots, M)} > \frac{p_{\ell_0}^n}{p_{\ell_1}^n},$$

And so the first order conditions of I 's problem cannot be satisfied if the consumption of ℓ_1 is positive in the limit. Thus, it must hold that

$$\left\{ (x_{\ell_1}^I(p^n, \pi(p^n, x(\cdot)))) \right\}_n \rightarrow 0,$$

that is, individual I 's equilibrium consumption of every good whose price goes doesn't go to zero converges to zero.

Therefore, individual I exhausts his budget (due to local nonsatiation), has a budget that converges to a positive value, and spends it entirely on goods whose prices converge to zero. This implies that there exists $\ell \in L_0$ such that

$$\left\{ (x_{\ell}^I(p^n, \pi(p^n, x(\cdot)))) \right\}_n \rightarrow \infty.$$

And since aggregate demand for every good is weakly bigger than individual I 's demand for any good, it follows that

$$\left\{ \max_k (x_k(p^n, \pi(p^n, x(\cdot)))) \right\}_n \rightarrow \infty,$$

a contradiction.

Since demands by consumers $i < I$ cannot be negative, and supply levels are bounded above, we have

$$\left\{ \max_k (z_k(p^n)) \right\}_n \rightarrow \infty. \tag{88}$$

It follows by Mas-Colell et al. (1995), Proposition 17.C.1 that $\exists \bar{p} \gg 0 : z(\bar{p}) = 0$. By definition, \bar{p} is a WES.

Given \mathcal{E} and $x(\cdot)$ as above, consider $(\mathcal{E}^{(n)}, x^{(n)}(\cdot))$. Aggregate excess demand in the replicated setting equals

$$z^{(n)}(p) = nx(p, \pi^{(n)}(p, x^{(n)}(\cdot))/n) - ns(p), \quad (89)$$

where aggregate supply in the replicated economy equals $ns(p)$ because the endowments and production capacities have both been multiplied by n , and where

$$\pi^{(n)}(p, x^{(n)}(\cdot)) \triangleq \pi : x^{(n)}(p, \pi) = ns(p) \quad (90)$$

$$\begin{aligned} &= \pi : x(p, \pi/n) = s(p) \\ &= n(\pi : x(p, \pi) = s(p)) \\ &= n\pi(p, x(\cdot)). \end{aligned} \quad (91)$$

Substituting (91) into (89), we have $z^{(n)}(p) = nz(p)$. It follows immediately that p is a WES of $(\mathcal{E}, x(\cdot))$ iff it is a WES of $(\mathcal{E}^{(n)}, x^{(n)}(\cdot))$, for any $n \geq 1$.

Finally, because $z(p)$ is h.o.d. 0 in p , any positive rescaling of \bar{p} is also a WES. In particular, $\hat{\bar{p}}$ is a WES. Thus $\hat{\bar{p}}$ is a normalized WES.

Conversely, if $\hat{\bar{p}}$ is a normalized WES, any positive rescaling of $\hat{\bar{p}}$, including \bar{p} , is a WES.

A.2 Proof of Proposition 2

The proof follows the structure of that of Proposition 17.D.5 of Mas-Colell et al. (1995).

Given \mathcal{E} , let $x^I(p, \pi, \omega^I)$ denote the unique admissible demand function for I in $\mathcal{E}(\omega^I)$. Also, given $(\{x^i(\cdot)\}_{i=1}^{I-1}, \bar{x}^I(\cdot)) \in \{\mathbb{A}^i(\mathcal{E})\}$, observe that, for all ω^I , $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E}(\omega^I)) \forall i < I$. Let

$$\hat{z}(\hat{p}, \omega^I) \triangleq \mathcal{I}[x^{-I}(\hat{p}, \pi(\hat{p}, x(\cdot))) + \bar{x}^I(\hat{p}, \pi(\hat{p}, x(\cdot)), \omega^I) - \hat{\omega}^{-I} - \hat{\omega}^I - y(\hat{p})]. \quad (92)$$

We will first show that $\text{Rank}(J_{\hat{z}, \omega^I}(\hat{p}, \omega^I)) = L - 1$ for all (\hat{p}, ω^I) with $s(\hat{p}) \gg 0$ and $\omega^I \gg 0$.

Given $\hat{p}, \omega^I \gg 0$, for every $\ell < L$ consider the marginal change to ω^I

$$\partial_\ell \omega^I \triangleq \hat{p}_\ell e_L - e_\ell. \quad (93)$$

Because $p \cdot \partial_\ell \omega^I = 0$, endowment-shift $\partial_\ell \omega^I$ will leave I 's wealth unchanged at prices p , and thus, fixing p , I 's demands \underline{x}^I unchanged at any given profit level. Thus, fixing p , $\partial_\ell \omega^I$ induces no shift to the profit level:

$$\begin{aligned} p \cdot (x^{-I}(p, \pi) + \underline{x}^I(p, \pi, \omega^I)) &= p \cdot (\omega + y(p)) \\ \implies p \cdot (x^{-I}(p, \pi) + \underline{x}^I(p, \pi, \omega^I + \partial_\ell \omega^I)) &= p \cdot (\omega + \partial_\ell \omega^I + y(p)). \end{aligned} \quad (94)$$

With $\partial_\ell \omega^I$ inducing no change to demand—or production, since p is fixed and y is a function of p —the change it induces to \hat{z} is precisely the first $L - 1$ entries of $-\partial_\ell \omega^I$. This is the $(L - 1)$ -length vector with a 1 in place $\ell < L$ and zeroes elsewhere. This space of marginal changes to \hat{z} —which can be induced by marginal changes to ω^I , holding \hat{p} fixed—spans (indeed is the canonical basis of) \mathbb{R}^{L-1} .

Thus $\text{Rank}(J_{\hat{z}, \omega^I}(\cdot)) = L - 1$. And thus, as long as $J_{\hat{z}, \hat{p}}(\hat{p}, \omega^I)$ is defined for some $\hat{p}, \omega^I \gg 0$, we have $\text{Rank}(J_{\hat{z}, \omega^I}(\hat{p}, \omega^I)) = L - 1$.

By assumption, $J_s(p)$ is defined for all p with $s(p) \gg 0$. Furthermore, $x(p, \pi)$ is differentiable in p and π . To show that $\chi(p) (= x(p, \pi(p, x(\cdot))))$ is differentiable in p around a $\bar{p} : s(\bar{p}) \gg 0$, we will now show that $\pi(p, x(\cdot))$ is differentiable in p around such a \bar{p} .

Let

$$h(p, \pi) \triangleq p \cdot (x(p, \pi) - s(p)). \quad (95)$$

Because $s(\cdot)$ and $x(\cdot)$ are \mathcal{C}^1 around (\bar{p}, π) for any π , and because the composition of \mathcal{C}^1 functions is \mathcal{C}^1 , $h(p, \pi)$ is \mathcal{C}^1 around (p, π) . By construction of the profit function, $h(p, \pi) = 0$. Also, because $p \cdot x(p, \pi)$ is increasing in π for any admissible $x(\cdot)$, $\partial h / \partial \pi$ is everywhere nonzero. Thus, by the implicit function theorem (IFT), the function $\pi(p, x(\cdot))$ such that $h(p, \pi(p, x(\cdot))) = 0$ for p near \bar{p} is \mathcal{C}^1 on p near \bar{p} .

$J_{\hat{z}, \hat{p}}(\hat{p}, \omega^I)$ is thus defined for all p with $s(p) \gg 0$. Mas-Colell et al. (1995), Proposition 17.D.3 then tells us that, for almost every—i.e. all but a measure-zero set of— ω^I , we have $\text{Rank}(J_{\hat{z}, \hat{p}}(\hat{p}, \omega^I)) = L - 1$ whenever $\hat{z}(\hat{p}, \omega^I) = 0$. That is, any normalized WES \hat{p} of $(\mathcal{E}(\omega^I), (\{x^i(\cdot)\}, \underline{x}(\cdot, \omega^I)))$ with $s(\hat{p}) \gg 0$ is regular. By Proposition 1, any such WES is regular as well.

Finally, replication of an economy and demand function profile by n multiplies the Jacobian of the corresponding normalized excess demand function, at any given \hat{p} , by n . A matrix is nonsingular iff it remains nonsingular after multiplication by a nonzero constant. Therefore regularity is preserved under replication.

A.3 Proof of Proposition 3

Let \bar{p} be a regular WES of economy \mathcal{E} and locally \mathcal{C}^1 demand functions $\{\bar{x}^i(\cdot)\} \in \mathbb{A}^i(\mathcal{E})$. Choose $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E})$ that is \mathcal{C}^1 around $(\bar{p}, \bar{\pi})$.

Let

$$h(\pi, \hat{p}, \alpha) \triangleq \dot{\hat{p}} \cdot (\bar{x}(\dot{\hat{p}}, \pi) + \alpha(x^i(\dot{\hat{p}}, \pi) - \bar{x}^i(\dot{\hat{p}}, \pi)) - s(\dot{\hat{p}})). \quad (96)$$

Because $s(\cdot)$, $x^i(\cdot)$, and each $\bar{x}^i(\cdot)$ are \mathcal{C}^1 around $(\dot{\hat{p}}, \bar{\pi}/p_L)$, and because the composition of \mathcal{C}^1 functions is \mathcal{C}^1 , $h(\cdot)$ is \mathcal{C}^1 around $(\dot{\hat{p}}, \bar{\pi}/p_L)$.

By construction of the profit function, $h(\pi(\dot{\hat{p}}, \bar{x}(\cdot)), \hat{p}, 0) = 0$. Also, because $p \cdot x(p, \pi)$ is increasing in π for any admissible $x(\cdot)$, $\partial h / \partial \pi$ is everywhere nonzero. Thus, by the IFT, there is a unique, \mathcal{C}^1 function, which we will denote $\hat{\pi}(\hat{p}, \alpha)$, such that $\hat{\pi}(\hat{p}, 0) = \pi(\dot{\hat{p}}, \bar{x}(\cdot))$ and $h(\hat{\pi}(\hat{p}, \alpha), \hat{p}, \alpha) = 0$ for all (\hat{p}, α) near $(\dot{\hat{p}}, 0)$.

Now, let

$$\begin{aligned} \hat{z}(\hat{p}, \alpha) \triangleq & \mathcal{I}[\bar{x}(\dot{\hat{p}}, \hat{\pi}(\hat{p}, \alpha)) \\ & + \alpha(x^i(\dot{\hat{p}}, \hat{\pi}(\hat{p}, \alpha)) - \bar{x}^i(\dot{\hat{p}}, \hat{\pi}(\hat{p}, \alpha))) - s(\dot{\hat{p}})]. \end{aligned} \quad (97)$$

Because $s(\cdot)$, $x^i(\cdot)$, and each $\bar{x}^i(\cdot)$ are locally \mathcal{C}^1 , and $\hat{\pi}(\cdot)$ is \mathcal{C}^1 , it follows that $\hat{z}(\cdot)$ is locally \mathcal{C}^1 .

Because \bar{p} is a WES of $(\mathcal{E}, x(\cdot))$, $\hat{z}(\hat{p}, 0) = 0$. Also, because \bar{p} is a regular WES, the Jacobian of $\hat{z}(\cdot)$ is nonsingular at $(\hat{p}, 0)$. Thus, by the IFT, there exist $\epsilon_1, \epsilon_2 > 0$ such that, for every $\alpha \in (-\epsilon_1, \epsilon_1)$, there is a unique $\hat{p} \in N_{\epsilon_2}(\hat{p})$ such that $\hat{z}(\hat{p}, \alpha) = 0$. Furthermore, defining

$$\hat{g}(\alpha) \triangleq \hat{p} : \hat{z}(\hat{p}, \alpha) = 0, \quad \alpha \in (-\epsilon_1, \epsilon_1), \quad (98)$$

$\hat{g}(\cdot)$ is \mathcal{C}^1 .

Thus

$$\hat{z}(\hat{g}(1/n), 1/n) = 0 \quad \forall n \geq \underline{n} \triangleq \lfloor 1/\epsilon_1 \rfloor + 1. \quad (99)$$

It follows that $\hat{g}(1/n)$ is a normalized WES, and $(\bar{p}_L \hat{g}(1/n), \bar{p}_L)$ is a WES, of economy \mathcal{E} and aggregate demand function

$$x(\cdot) \triangleq \bar{x}(\cdot) + (x^i(\cdot) - \bar{x}^i(\cdot))/n. \quad (100)$$

It also follows that $\hat{g}(1/n)$ is the unique normalized WES in $N_{\epsilon_2}(\hat{p})$.

Finally, because $\hat{z}(\cdot)$ is locally \mathcal{C}^1 , and because its Jacobian determinant is nonzero at $(\hat{p}, 0)$, there is an $\epsilon_3 > 0$ such that, for all $(\hat{p}, \alpha) \in N_{\epsilon_3}((\hat{p}, 0))$, its Jacobian determinant is nonzero at (\hat{p}, α) . Therefore, as long as we pick ϵ_1 and ϵ_2 small enough that

$$\sqrt{\epsilon_1^2 + \epsilon_2^2} \leq \epsilon_3, \quad (101)$$

we can guarantee that, as long as $n \geq \lfloor 1/\epsilon_1 \rfloor + 1$, $\hat{g}(1/n)$ is not only a WES but a regular WES of $(\mathcal{E}, x(\cdot))$.

It then follows from Proposition 1 and the definition of n -replicated utility that, as long as $n \geq \lfloor 1/\epsilon_1 \rfloor + 1$, $\hat{g}(1/n)$ is the unique normalized WES in $N_{\epsilon_2}(\hat{\bar{p}})$, and from Proposition 2 that it is a regular normalized WES, of $(\mathcal{E}^{(n)}, \bar{x}^{-i(n)}(\cdot) + x^{i(n)}(\cdot))$.

A.4 Proof of Proposition 4

Let \bar{p} be a WES of economy \mathcal{E} and locally \mathcal{C}^1 demand function profile $\{\bar{x}^i(\cdot)\} \in \{\mathbb{A}^i(\mathcal{E})\}$.

Let \underline{n} be large enough that $\mathcal{E}^{(n)}$ is large with respect to $\{\bar{x}^i(\cdot)\}^{(n)}$ and $\hat{\bar{p}}$. Then, given $n \geq \underline{n}$, consider the change in $u_{\bar{p}}^{i(n)}$ that i achieves by deviating to locally \mathcal{C}^1 demand function $x^i(\cdot) \in \mathbb{A}^i(\mathcal{E})$:

$$\begin{aligned} & v^i \left(x^{i(n)} \left(p_{\bar{p}}^{(n)}(x^i(\cdot)), \pi^{(n)}(p_{\bar{p}}^{(n)}(x^i(\cdot)), \bar{x}^{-i(n)}(\cdot) + x^{i(n)}(\cdot)) \right) \right) + w^i \left(ns \left(p_{\bar{p}}^{(n)}(x^i(\cdot)) \right) \right) \\ & - v^i \left(\bar{x}^{i(n)}(\bar{p}, \pi^{(n)}(\bar{p}, \bar{x}^{(n)}(\cdot))) \right) - w^i(ns(\bar{p})). \end{aligned} \quad (102)$$

In steps, we will take the limit of (102) as $n \rightarrow \infty$ and determine when the expression is nonpositive for any admissible choice of $x^i(\cdot)$.

First, by definitions (24) and (29), and by the assumption that $w^i(\cdot)$ is CRS, (102) equals

$$\begin{aligned} & v^i \left(x^i(p_{\bar{p}}^{(n)}(x^i(\cdot)), \pi(p_{\bar{p}}^{(n)}(x^i(\cdot)), \bar{x}(\cdot) + (x^i(\cdot) - \bar{x}^i(\cdot))/n)) \right) - v^i(\bar{x}^i(\bar{p}, \bar{\pi})) \\ & + nw^i(s(p_{\bar{p}}^{(n)}(x^i(\cdot)))) - nw^i(s(\bar{p})). \end{aligned} \quad (103)$$

By Proposition 3 and definition (39) of $p_{\bar{p}}^{(n)}(\cdot)$, we have

$$p_{\bar{p}}^{(n)}(x^i(\cdot)) = g(1/n) \triangleq (\hat{g}(1/n), 1), \quad (104)$$

where $\hat{g}(\cdot)$ is defined as in (98). Because $g(\alpha)$ is continuous and equals $\hat{\bar{p}}$ at $\alpha = 0$, and because $x^i(p, \pi)$ is h.o.d. 0 and $\pi(p, x(\cdot))$ is h.o.d. 1 in p , the limit as $n \rightarrow \infty$ of the first term of (103) equals

$$v^i(x^i(\bar{p}, \bar{\pi})). \quad (105)$$

Substituting (104) into the third term of (103), and replacing n with $1/\alpha$ (where $\alpha \triangleq 1/n$), the third and fourth terms equal

$$\frac{w^i(s(g(\alpha))) - w^i(s(\bar{p}))}{\alpha}. \quad (106)$$

Because $w^i(\cdot)$, $s(\cdot)$, and $g(\cdot)$ are differentiable with $s(g(0)) = s(\bar{p})$, the limit of (106) as $\alpha \rightarrow 0$ equals

$$\frac{\partial}{\partial \alpha} [w^i(s(g(\alpha)))]_{\alpha=0}, \quad (107)$$

which, by the chain rule, equals

$$\nabla w^i(s(\bar{p})) \cdot J_s(\bar{p}) \nabla g(0). \quad (108)$$

The first and second of these partial derivatives is given directly by the functions $w^i(\cdot)$ and $s(\cdot)$. We will now find the third: $\nabla g(0) = (\nabla \hat{g}(0), 0)$.

Recall the construction of $\hat{g}(\alpha)$ in (96)–(98). By the IFT,

$$\nabla \hat{g}(0) = - \left(J_{\hat{z}, \hat{p}}(\hat{p}, 0) \right)^{-1} J_{\hat{z}, \alpha}(\hat{p}, 0), \quad (109)$$

where $\hat{z}(\hat{p}, \alpha)$ is defined as in (97) and $J_{\hat{z}, \hat{p}}(\hat{p}, 0)$ and $J_{\hat{z}, \alpha}(\hat{p}, 0)$ are the Jacobians of $\hat{z}(\hat{p}, \alpha)$ with respect to \hat{p} and α , respectively, evaluated at $(\hat{p}, 0)$.

Element ℓ , k of $J_{\hat{z}, \hat{p}}(\hat{p}, 0)$ (defined for $\ell, k < L$) equals

$$\frac{\partial \bar{x}_\ell(\hat{p}, \hat{\pi})}{\partial p_k} + \frac{\partial \bar{x}_\ell(\hat{p}, \hat{\pi})}{\partial \pi} \frac{\partial \hat{\pi}(\hat{p}, 0)}{\partial \hat{p}_k} - \frac{\partial s_\ell(\hat{p})}{\partial p_k}. \quad (110)$$

Element ℓ ($< L$) of $J_{\hat{z}, \alpha}(\hat{p}, 0)$ equals

$$\frac{\partial \bar{x}_\ell(\hat{p}, \hat{\pi})}{\partial \pi} \frac{\partial \hat{\pi}(\hat{p}, 0)}{\partial \alpha} + \Delta x_\ell^i, \quad (111)$$

where

$$\Delta x^i \triangleq x^i(\bar{p}, \bar{\pi}) - \bar{x}^i(\bar{p}, \bar{\pi}). \quad (112)$$

Likewise, recalling the construction of $\hat{\pi}(\hat{p}, \alpha)$ following (96), and letting

$$\hat{\pi} \triangleq \pi(\hat{p}, \bar{x}(\cdot)) \quad (= \bar{\pi}/\bar{p}_L, = \hat{\pi}(\hat{p}, 0)), \quad (113)$$

the IFT gives us

$$\frac{\partial \hat{\pi}(\hat{p}, 0)}{\partial \hat{p}_k} = - \frac{1}{\bar{p} \cdot \hat{\delta}} \sum_{\ell=1}^L \bar{p}_\ell \left(\frac{\partial \bar{x}_\ell(\hat{p}, \hat{\pi})}{\partial p_k} - \frac{\partial s_\ell(\hat{p})}{\partial p_k} \right) \quad (k < L), \quad (114)$$

$$\frac{\partial \hat{\pi}(\hat{p}, 0)}{\partial \alpha} = - \frac{1}{\bar{p} \cdot \hat{\delta}} \bar{p} \cdot \Delta x^i, \quad (115)$$

where

$$\bar{\delta} \triangleq \nabla_{\pi} \bar{x}(\bar{p}, \bar{\pi}), \quad (116)$$

$$\hat{\delta} \triangleq \nabla_{\pi} \bar{x}(\hat{p}, \hat{\pi}) \quad (= \bar{p}_L \bar{\delta}). \quad (117)$$

Note that the denominator terms $\bar{p} \cdot \bar{\delta}$ must be nonzero by the admissibility of $\bar{x}(\cdot)$.

Substituting (114) and (115) into (110) and (111), expression (109) can be rewritten

$$\nabla \hat{g}(0) = \left[M\mathcal{I}(\hat{\delta} \times \bar{p} - (\bar{p} \cdot \hat{\delta})I_L) \right] \Delta x^i, \quad (118)$$

and so

$$\nabla g(0) = \left[M\mathcal{I}(\hat{\delta} \times \bar{p} - (\bar{p} \cdot \hat{\delta})I_L) \right]_{\mathbb{0}_L^T} \Delta x^i, \quad (119)$$

where

$$M \triangleq \left(\mathcal{I} \left((\bar{p} \cdot \bar{\delta}) (J_{\bar{x},p}(\hat{p}, \hat{\pi}) - J_s(\hat{p})) - D(\bar{\delta}) (J_{\bar{x},p}(\hat{p}, \hat{\pi}) - J_s(\hat{p})) \bar{p} \times \mathbb{1}_L \right) \mathcal{I}^T \right)^{-1}. \quad (120)$$

Let G denote the matrix coefficient on Δx^i in (119). We will now provide an alternative, and in some ways simpler, characterization of G .

Define

$$\underline{\pi}(p, \alpha) \triangleq \pi(p, \bar{x}(\cdot) + \alpha(x^i(\cdot) - \bar{x}^i(\cdot))) \quad (= p_L \hat{\pi}(\hat{p}, \alpha)), \quad (121)$$

$$\underline{x}(p, \alpha) \triangleq x(p, \underline{\pi}(p, \alpha)), \quad (122)$$

and define $\underline{x}^i(p, \alpha)$ likewise. Since $\hat{\pi}(\hat{p}, \alpha)$, as defined following (96), is differentiable and $\pi(p, x(\cdot))$ is h.o.d. 1 in p , $\underline{\pi}(\cdot)$ is differentiable. Thus $\underline{x}(p, \alpha)$ and $\underline{x}^i(p, \alpha)$ are differentiable.

Now observe that, for all $\alpha \leq 1/\underline{n}$, the following hold exactly:

$$\underline{\bar{x}}(g(\alpha), \alpha) + \alpha(\underline{x}^i(g(\alpha), \alpha) - \underline{\bar{x}}^i(g(\alpha), \alpha)) = s(g(\alpha)), \quad (123)$$

$$\underline{\bar{x}}(\bar{p}, 0) = s(\bar{p}) \quad (124)$$

$$\implies s(g(\alpha)) - s(\bar{p}) - (\underline{\bar{x}}(\bar{p}, 0) - \underline{\bar{x}}(g(\alpha), \alpha)) = \alpha(\underline{x}^i(g(\alpha), \alpha) - \underline{\bar{x}}^i(g(\alpha), \alpha)). \quad (125)$$

Dividing both sides of (125) by α and taking the limit as $\alpha \rightarrow 0$, we have

$$-J_z(\bar{p}) \nabla g(0) + \bar{\delta} \left(\frac{\partial \underline{\pi}(\bar{p}, 0)}{\partial \alpha} \right) = \Delta x^i. \quad (126)$$

Because $\nabla g(0) = G\Delta x^i$ and $\partial \underline{\pi}(\bar{p}, 0)/\partial \alpha = \partial \hat{\underline{\pi}}(\hat{\bar{p}}, 0)/\partial \alpha$, we then have

$$-J_z(\bar{p})G\Delta x^i - \frac{p \cdot \Delta x^i}{p \cdot \bar{\delta}} \bar{\delta} = \Delta x^i. \quad (127)$$

Thus, for Δx^i satisfying

$$\bar{p} \cdot \Delta x^i = 0, \quad (128)$$

we have

$$-J_z(\bar{p})G\Delta x^i = \Delta x^i. \quad (129)$$

Observe that $J_s(\bar{p})\bar{p} = 0$. If prices all rise in proportion to their current levels, then the price vector has simply been rescaled, and because $s(p)$ is h.o.d. 0, supply levels will not change. $J_{\underline{x}}(\bar{p})\bar{p} = 0$ likewise, by the definition of \underline{x} and the fact that $\chi(p)$ is h.o.d. 0. So it follows from the definition (31) of $z(p)$ that

$$J_z(\bar{p})\bar{p} = 0. \quad (130)$$

Also, for any Δp *not* proportional to \bar{p} , we have $\Delta \hat{p} = \hat{p} - \hat{\bar{p}} \neq 0_{L-1}$. Because \bar{p} is a regular WES, $J_{\hat{z}}(\hat{\bar{p}})$ is of full rank, so $J_{\hat{z}}(\hat{\bar{p}})\Delta \hat{p} \neq 0_{L-1}$. It follows that any marginal price-change Δp not proportional to \bar{p} induces a change to excess demands, and thus that $J_z(\bar{p})\Delta p \neq 0_L$. Thus

$$\text{Rank}(J_z(\bar{p})) = L - 1. \quad (131)$$

Returning to (129), we can now conclude that G is a generalized inverse of $-J_z(\bar{p})$. Furthermore, from (127), at $\Delta x^i = -\bar{\delta}$ we have

$$-J_z(\bar{p})G\bar{\delta} + \bar{\delta} = \bar{\delta}. \quad (132)$$

So $G\bar{\delta}$ either equals 0_L or is proportional to \bar{p} . But we know that the bottom row of G consists of zeroes, so the last entry of $G\bar{\delta}$ equals 0. So

$$G\bar{\delta} = 0_L. \quad (133)$$

We will now construct the unique generalized inverse G of $J_z(\bar{p})$ whose bottom row consists of zeroes and for which $G\bar{\delta} = 0_L$.

Let UNV^T denote a singular value decomposition of $J_z(\bar{p})$ with $N_{\ell\ell} \neq 0$ for $\ell < L$ and $N_{LL} = 0$. Because G is a generalized inverse of $J_z(\bar{p})$, we must by (131) have

$$G = V \begin{bmatrix} N_1^{-1} & A \\ B & C \end{bmatrix} U^T, \quad (134)$$

where N_1 is the $(L-1) \times (L-1)$ principal submatrix of N and A , B , and C are $(L-1) \times 1$, $1 \times (L-1)$, and 1×1 respectively.

Since V is invertible, (133) reduces to

$$\begin{bmatrix} N_1^{-1} & A \\ B & C \end{bmatrix} \bar{\bar{\delta}} = \mathbb{0}_L \quad (135)$$

$$\implies A_\ell = -\frac{\bar{\bar{\delta}}_\ell}{\bar{\bar{\delta}}_L} \frac{1}{N_{\ell\ell}}, \quad \ell < L, \text{ and} \quad (136)$$

$$C = -\frac{1}{\bar{\bar{\delta}}_L} \sum_{\ell=1}^{L-1} B_\ell \bar{\bar{\delta}}_\ell, \quad (137)$$

where $\bar{\bar{\delta}} \triangleq U^T \bar{\delta}$.

We will now impose the constraint that the bottom row of G consists of zeroes. Since U^T is invertible, the bottom row of G consists of zeroes iff the bottom row of

$$V \begin{bmatrix} N_1^{-1} & A \\ B & C \end{bmatrix} \quad (138)$$

consists of zeroes. This in turn implies

$$B_\ell = -\frac{V_{L\ell}}{V_{LL}} \frac{1}{N_{\ell\ell}}, \quad \ell < L. \quad (139)$$

By (137), this also gives us C . G is thus fully constructed.

Substituting $G\Delta x^i$ for $\nabla g(0)$ into (108), we have $\psi^i \cdot \Delta x^i$, where

$$\psi^i = (J_s(\bar{p})G)^T \nabla w^i(s(\bar{p})). \quad (140)$$

Thus the limit of (102) as $n \rightarrow \infty$ equals

$$(v^i(x^i(\bar{p}, \bar{\pi})) + \psi^i \cdot x^i(\bar{p}, \bar{\pi})) - (v^i(\bar{x}^i(\bar{p}, \bar{\pi})) + \psi^i \cdot \bar{x}^i(\bar{p}, \bar{\pi})). \quad (141)$$

And thus $(\bar{p}, \{\bar{x}^i(\cdot)\})$ is a CESE iff $\bar{x}^i(\bar{p}, \bar{\pi})$ maximizes $v^i(x^i(\bar{p}, \bar{\pi})) + \psi^i \cdot x^i(\bar{p}, \bar{\pi})$, among feasible $x^i(\bar{p}, \bar{\pi})$, for all i .

Finally, we can express (140) in elasticity terms. For $\Delta x^i : \bar{p} \cdot \Delta x^i = 0$,

$$-J_z(\bar{p})G\Delta x^i = \Delta x^i \quad (142)$$

$$\implies D(s(\bar{p}))(\sigma(\bar{p}) - \varepsilon(\bar{p}, \bar{x}(\cdot)))D(\bar{p})^{-1}G\Delta x^i = \Delta x^i \quad (143)$$

$$\implies (\sigma(\bar{p}) - \varepsilon(\bar{p}, \bar{x}(\cdot)))\phi D(s(\bar{p}))^{-1}\Delta x^i = D(s(\bar{p}))^{-1}\Delta x^i, \quad (144)$$

where

$$\phi \triangleq D(\bar{p})^{-1}GD(s(\bar{p})) \quad (145)$$

is the generalized inverse of $(\sigma(\bar{p}) - \varepsilon(\bar{p}, \bar{x}(\cdot)))$ with $\phi D(s(\bar{p}))^{-1} \bar{\delta} = 0$ and whose bottom row consists of zeroes. Substituting

$$G = D(\bar{p}) \phi D(s(\bar{p}))^{-1}, \quad (146)$$

$$J_s(\bar{p}) = D(s(\bar{p})) \sigma D(\bar{p})^{-1} \quad (147)$$

into (140), we have

$$\psi^i = (D(s(\bar{p})) \sigma D(\bar{p}) \phi D(s(\bar{p}))^{-1})^T \nabla w^i(s(\bar{p})). \quad (148)$$

The diagonal matrices cancel.

A.5 Proof of Proposition 8

Suppose the utility functions and endowments of an economy \mathcal{E} satisfy assumptions (73)–(75) with respect to *some* bounds $\{\bar{\psi}^i\}$, and choose an externality matrix $\bar{\psi}$ with $|\psi_\ell^i| < \bar{\psi}^i \forall i, \ell < L$ and $\bar{\psi}_L^i = 0 \forall i$. (We will show that there is a WES \bar{p} of $(\mathcal{E}, x_{[\bar{\psi}]}(\cdot))$ and that $\psi(\bar{p}, \bar{\psi})$ lies within bounds independent of the $\{t_\ell^i\}$ and $\{\omega_\ell^i\}$. These bounds will constitute those used in the statement of the proposition.)

For any such $\bar{\psi}$, each i will have quasi-utility function

$$\tilde{u}^i(x^i, \bar{\psi}^i) = \sum_{\ell=1}^{L-1} \left(\tilde{v}_\ell^i(x_\ell^i) + t_\ell^i x_\ell^i + \bar{\psi}_\ell^i x_\ell^i \right) + t_L^i x_L^i. \quad (149)$$

Our assumptions guarantee that the resulting demand function $x_{[\bar{\psi}]}^i(\cdot)$ exists and is admissible for all i (as noted in Footnote 8). By Proposition 1, there is a WES $\bar{p} \gg 0$ of $(\mathcal{E}, x_{[\bar{\psi}]}(\cdot))$.

Furthermore, because $\bar{\psi}_L^i = 0 \forall i$, individuals always exhaust their budgets; so profits equal $p \cdot y(p)$ for all p . By the first-order conditions on implicit demand, therefore, for any p at which $\chi_{[\bar{\psi}]\ell}^i(p) > 0 \forall \ell$, we have

$$\frac{t_L^i}{p_L} = \frac{\tilde{v}_\ell^{i'}(\chi_{[\bar{\psi}]\ell}^i(p)) + t_\ell^i + \bar{\psi}_\ell^i}{p_\ell} \quad (150)$$

$$\implies \chi_{[\bar{\psi}]\ell}^i(p) = \tilde{v}_\ell^{i'-1} \left(\frac{p_\ell}{p_L} t_L^i - t_\ell^i - \bar{\psi}_\ell^i \right), \quad \ell < L; \quad (151)$$

$$\chi_{[\bar{\psi}]L}^i(p) = \frac{1}{p_L} \left(p \cdot \omega^i - \theta^i p \cdot y(p) - \sum_{\ell=1}^{L-1} \chi_{[\bar{\psi}]\ell}^i(p) \right). \quad (152)$$

Note that p must therefore have the argument of $\tilde{v}_\ell^{i'-1}(\cdot)$ in (151) positive for all $\ell < L$ (as it does if $p \in P$, by (74)). And for all $\ell < L$, $\tilde{v}_\ell^{i'}(\cdot)$ is one-to-one on $\mathbb{R}_{>0}$ (by the upper and lower Inada conditions), \mathcal{C}^1 , and has a derivative that is everywhere

nonzero. Thus, at such a p , $\tilde{v}_\ell^{i'-1}(\cdot)$ is defined and also \mathcal{C}^1 , by the inverse function theorem. So from (151), $\chi_{[\psi]\ell}^i(p)$ is \mathcal{C}^1 in p and ψ . From (152), $\chi_{[\psi]L}^i(p)$ is as well.

The lower Inada conditions on the $\tilde{v}_\ell^i(\cdot)$, and the strict concavity, guarantee that, for all $p \gg 0$, $\chi_{[\psi]\ell}^i(p) > 0 \forall \ell < L$. It follows that a WES \bar{p} must have $s_\ell(\bar{p}) > 0 \forall \ell < L$.

A WES \bar{p} must thus have $s_L(\bar{p}) > 0$ as well. To see this, first recall that profit-maximization requires $F(y(\bar{p})) = 0$. Because $F(\cdot)$ is differentiable, strictly increasing, and strictly convex, $J_F(y)$ not only exists but is invertible for all y . Then because $F(\cdot)$ is \mathcal{C}^1 , it follows from the IFT that, for any $\ell < L$, there is an $\bar{\epsilon}_\ell > 0$ such that, for all $\epsilon_\ell \in (-\bar{\epsilon}_\ell, \bar{\epsilon}_\ell)$,

$$g(\epsilon_\ell) \triangleq \epsilon_L : F(y(\bar{p}) - \epsilon_\ell e_\ell + \epsilon_L e_L) = 0 \quad (153)$$

is defined and differentiable, with

$$g'(0) = (\nabla F(y(\bar{p})))_\ell / (\nabla F(y(\bar{p})))_L. \quad (154)$$

So, choosing a sufficiently small $\epsilon_\ell < s_\ell(\bar{p})$, it would be feasible to cut the supply of ℓ by ϵ_ℓ , and increase the supply of L by approximately $g'(0) \epsilon_\ell$, without changing the supply of other goods.

Next, we will show that there exists an i (indeed, all i) for which

$$\frac{t_L^i}{\partial \tilde{u}^i(x^i, \bar{\psi}) / \partial x_\ell^i} > \frac{1}{g'(0)} \quad (155)$$

when $x_L^i = 0$ and x_ℓ^i is compatible with equilibrium for $\ell < L$. Because, by assumption, ω^i satisfies

$$\begin{aligned} t_L^i &> \frac{\tilde{v}_\ell^{i'}(\omega_\ell^i) + t_\ell^i + \bar{\psi}_\ell^i}{\underline{p}_\ell} \\ \iff \omega_\ell^i &> \tilde{v}_\ell^{i'-1}(\underline{p}_\ell t_L^i - t_\ell^i - \bar{\psi}_\ell^i) \quad \forall \ell < L \end{aligned} \quad (156)$$

(recalling that the argument of $\tilde{v}_\ell^{i'-1}(\cdot)$ is positive by (74)), i prefers marginal purchases of L to marginal purchases of any $\ell < L$, given a price ratio in P , at the endowment point. Given $x_L^i = 0$ and budget exhaustion (which follows, again, from $\bar{\psi}_L^i = 0 \forall i$), we must have $x_\ell^i \geq \omega_\ell^i$ for some $\ell < L$. Thus, since $\tilde{v}_\ell^{i'}(\cdot)$ is concave,

$$t_L^i > \frac{\tilde{v}_\ell^{i'}(x_\ell^i) + t_\ell^i + \bar{\psi}_\ell^i}{\underline{p}_\ell} = \frac{1}{\underline{p}_\ell} \frac{\partial \tilde{u}^i(x^i, \bar{\psi}^i)}{\partial x_\ell^i}$$

for some ℓ . (155) then follows from the fact that $\underline{p}_\ell \leq g'(0)$ by definition.

Given budget exhaustion, the allocation induced by a WES \bar{p} must be efficient (with respect to the \tilde{u}^i), by the first welfare theorem. But if $x_L^i = s_L(\bar{p}) = 0$, there

is an $\ell < L$ such that \tilde{u}^i can be increased by shifting production marginally from ℓ to L , in a feasible direction, without affecting \tilde{u}^j for $j \neq i$. So $s(\bar{p}) \gg 0$ for any WES \bar{p} .

For any WES \bar{p} , $\hat{\bar{p}}$ lies in P . This follows from $s(\bar{p}) \gg 0$, as following (5).

Also, the WES of $(\mathcal{E}, x_{[\bar{\psi}]}(\cdot))$ is unique up to rescaling. This follows from the fact that, for each i , $x_{[\bar{\psi}]}(\cdot)$ maximizes a quasilinear (quasi-)utility function: see Hosoya (2022), Theorem 1. Hosoya assumes that utility in goods $\ell < L$ is nondecreasing, whereas we allow marginal utility in such goods to be negative at sufficiently large values of x_ℓ^i , because we may have $\psi_\ell^i < -t_\ell^i$. But because each i 's demands are identical to those that would obtain if marginal utility in each ℓ equaled 0 (rather than a negative number) at such large values of x_ℓ^i , Hosoya's result holds in our context.

There is thus a unique WES of $(\mathcal{E}, x_{[\bar{\psi}]}(\cdot))$ in P . Let us denote it simply by \bar{p} .

Because $\bar{p} \in P$, $\chi_{\bar{\psi}L}^i(\bar{p}) > 0 \forall i$. This follows from (156) as above.

We will now show that \bar{p} is regular.

Given $k, \ell < L$, it follows from (151) that $\partial \chi_{[\bar{\psi}]\ell}^i(\bar{p}) / \partial p_k < 0$ if $k = \ell$ and $= 0$ otherwise. So, letting

$$J_{\hat{\chi}_{[\bar{\psi}]}}(\hat{\bar{p}}) \triangleq \mathcal{I} J_{\chi_{[\bar{\psi}]}}(\bar{p}) \mathcal{I}^T \quad (157)$$

denote the upper-left $(L-1) \times (L-1)$ submatrix of $J_{\chi_{[\bar{\psi}]}}(\bar{p})$, $J_{\hat{\chi}_{[\bar{\psi}]}}(\hat{\bar{p}})q$ is, for any $q \in \mathbb{R}^{L-1}$, an $(L-1)$ -length vector whose entries have opposite signs to those of q . $J_{\hat{\chi}_{[\bar{\psi}]}}(\hat{\bar{p}})$ is thus strictly negative definite.

Our assumptions guarantee that $J_{\bar{y}}(p)$ is positive semidefinite for $p \in P$ (see Mas-Colell et al. (1995), Proposition 5.C.1 (vii)). They thus also guarantee that $J_s(p)$ is positive semidefinite for $p \in \tilde{P}$. $J_{\hat{s}}(\hat{\bar{p}}) \triangleq \mathcal{I} J_s(\bar{p}) \mathcal{I}^T$ is therefore also positive semidefinite:

$$q \cdot J_{\hat{s}}(\hat{\bar{p}})q = (q, 0)^T \cdot J_s(\bar{p})(q, 0)^T \leq 0 \quad \forall q \in \mathbb{R}^{L-1}. \quad (158)$$

$J_{\hat{z}_{[\bar{\psi}]}}(\hat{\bar{p}}) = J_{\hat{s}}(\hat{\bar{p}}) - J_{\hat{\chi}_{[\bar{\psi}]}}(\hat{\bar{p}})$ is thus positive definite, and therefore nonsingular.

As can be seen from (152), quasilinearity and the fact that $\chi_{[\bar{\psi}]L}^i(\bar{p}) > 0 \forall i$ imply $\bar{\delta} \triangleq \delta(\bar{p}, \bar{\psi}) = e_L$. Marginal income is spent entirely on L , whose price equals 1.

Because of the regularity of \bar{p} , $G(\bar{p}, \bar{\psi})$ is defined. Let us likewise denote it by \bar{G} , for simplicity.

Finally, because \bar{G} is defined, $\psi(\bar{p}, \bar{\psi})$ is defined. We will now find bounds on $\psi(\bar{p}, \bar{\psi})$.

Let \mathcal{S}^n denote the n -dimensional unit sphere. Observe that some $q \in \mathcal{S}^{L-2}$ is an $(L-1)$ -length vector with a norm of 1, and that $\tilde{q} \triangleq \mathcal{I}^T q$ is an L -length vector with a norm of 1 and a zero in the L^{th} place.

Given $p \in \tilde{P}$ and $\hat{q} \in \mathbb{R}^{L-1}$, let $\hat{\psi}(p, \hat{q})$ denote the I -vector with entry i equal to

$$\hat{\psi}^i(p, \hat{q}) \triangleq \nabla w^i(s(p)) \cdot J_s(p) \mathcal{I}^T \hat{q}. \quad (159)$$

In effect, $\hat{\psi}^i(p, \hat{q})$ is the size of the ethical externality, for i , resulting from a marginal price-change of \hat{q} that keeps the price of good L fixed, starting from normalized prices p .

Since $s(\cdot)$ is \mathcal{C}^1 and $w^i(\cdot)$ is \mathcal{C}^1 for all i , $\hat{\psi}(\cdot)$ is a composition of continuous functions and thus continuous. Therefore, given any (p, \hat{q}) for which $\hat{\psi}(p, \hat{q}) = \mathbf{0}_I$ and any $\underline{\psi} > 0$, there exists an $\epsilon > 0$ such that

$$\max(|\hat{\psi}(\tilde{p}, \tilde{\hat{q}})|) < \underline{\psi} \quad \forall (\tilde{p}, \tilde{\hat{q}}) \in \hat{N}_\epsilon(p, \hat{q}), \quad (160)$$

where $\hat{N}_\epsilon(p, \hat{q})$ denotes the ϵ -neighborhood around (p, \hat{q}) in $P \times \mathbb{R}^{L-1}$.

Fixing $\underline{\psi}$, denote the supremum of such ϵ for each (p, \hat{q}) with $\hat{\psi}(p, \hat{q}) = \mathbf{0}_I$ by $\epsilon(\underline{\psi}, p, \hat{q})$. Let

$$\hat{N}(\underline{\psi}) \triangleq \cup_{\{(p, \hat{q}) \in P \times \mathbb{R}^{L-1} \mid \hat{\psi}(p, \hat{q}) = \mathbf{0}_I\}} N_{\epsilon(\underline{\psi}, p, \hat{q})}(p, \hat{q}), \quad (161)$$

$$N(\underline{\psi}) \triangleq \left\{ \left(p, \frac{\hat{q}}{\|\hat{q}\|} \right) \mid (p, \hat{q}) \in \hat{N}(\underline{\psi}), \hat{q} \neq \mathbf{0}_{L-1} \right\}. \quad (162)$$

$N(\underline{\psi}) \subset P \times \mathcal{S}^{L-2}$. Let us now show that $N(\underline{\psi})$ is open in $P \times \mathcal{S}^{L-2}$.

For any $(p, q) \in N(\underline{\psi})$, we can find a $\hat{q} \in \mathbb{R}^{L-1} \setminus \{\mathbf{0}_{L-1}\}$ such that $(p, \hat{q}) \in \hat{N}(\underline{\psi})$ and

$$\frac{\hat{q}}{\|\hat{q}\|} = q. \quad (163)$$

As a union of open sets, $\hat{N}(\underline{\psi})$ is open in $P \times \mathbb{R}^{L-1}$; so there exists an $\hat{\epsilon} > 0$ such that

$$\hat{N}_{\hat{\epsilon}}(p, \hat{q}) \subset \hat{N}(\underline{\psi}). \quad (164)$$

Now let

$$\tilde{\epsilon} \triangleq \frac{\hat{\epsilon}}{\sqrt{1 + \|\hat{q}\|^2}} > 0. \quad (165)$$

Also, in general, let $N_\epsilon(p, q)$ denote the ϵ -neighborhood around (p, q) in $P \times \mathcal{S}^{L-2}$.

Now choose any $(\tilde{p}, \tilde{q}) \in N_{\tilde{\epsilon}}(p, q)$. We must have $\|\tilde{p} - p\| < \tilde{\epsilon}$ and $\|\tilde{q} - q\| < \tilde{\epsilon}$, by the triangle inequality. We must thus have

$$\left\| (\tilde{p}, \tilde{q} \|\hat{q}\|) - (p, q \|\hat{q}\|) \right\| < \sqrt{\tilde{\epsilon}^2 + (\|\hat{q}\| \tilde{\epsilon})^2} = \hat{\epsilon}, \quad (166)$$

also by the triangle inequality. By (163), $q||\hat{q}|| = \hat{q}$. (166) then tells us that $(\tilde{p}, \tilde{q}||\hat{q}||) \in \hat{N}_\varepsilon(p, \hat{q})$ —and so, by (164), that

$$(\tilde{p}, \tilde{q}||\hat{q}||) \in \hat{N}(\underline{\psi}). \quad (167)$$

Since $\tilde{q} \in \mathcal{S}^{L-2}$, $\tilde{q} \neq \mathbb{0}_{L-1}$; and since $\hat{q} \neq \mathbb{0}_{L-1}$ either, $\tilde{q}||\hat{q}|| \neq \mathbb{0}_{L-1}$. So

$$\left(\tilde{p}, \frac{\tilde{q}||\hat{q}||}{||\tilde{q}||\hat{q}||} \right) \in N(\underline{\psi}). \quad (168)$$

And since $\tilde{q} \in \mathcal{S}^{L-2}$, we have $||\tilde{q}|| = 1$, and thus

$$\frac{\tilde{q}||\hat{q}||}{||\tilde{q}||\hat{q}||} = \tilde{q}. \quad (169)$$

So $(\tilde{p}, \tilde{q}) \in N(\underline{\psi})$.

We have now shown that any $(p, q) \in N(\underline{\psi})$ is surrounded by an open set in $P \times \mathcal{S}^{L-2}$ also contained in $N(\underline{\psi})$. This completes the proof that $N(\underline{\psi})$ is open in $P \times \mathcal{S}^{L-2}$.

Since $P \times \mathcal{S}^{L-2}$ is compact and $N(\underline{\psi})$ is open,

$$\mathcal{Q}(\underline{\psi}) \triangleq (P \times \mathcal{S}^{L-2}) \setminus N(\underline{\psi}) \quad (170)$$

is compact.

Recall that $\psi_\ell^i(\bar{p}, \bar{\psi}) = (J_s(\bar{p})\bar{G}e_\ell)^T \nabla w^i(s(\bar{p}))$, and again, fix $\underline{\psi}$. If $(\bar{p}, \mathcal{I}\bar{G}e_\ell) \in \hat{N}(\underline{\psi})$, then $|\psi_\ell^i(\bar{p}, \bar{\psi})| \leq \underline{\psi} \forall i$.

If not, observe that $\check{q} \cdot J_s(p)\check{q}$ is continuous in p (since $s(\cdot)$ is \mathcal{C}^1) and in q . Also, recall that $\mathcal{Q}(\underline{\psi})$ is compact. Finally, observe that (at least for this particular choice of $\underline{\psi}$) $\mathcal{Q}(\underline{\psi})$ is nonempty, since it contains at least

$$\left(\bar{p}, \frac{\mathcal{I}\bar{G}e_\ell}{||\mathcal{I}\bar{G}e_\ell||} \right), \quad (171)$$

by the supposition that $(\bar{p}, \mathcal{I}\bar{G}e_\ell) \notin \hat{N}(\underline{\psi})$. By the extreme value theorem, we can thus define

$$m(\underline{\psi}) \triangleq \min_{(p,q) \in \mathcal{Q}(\underline{\psi})} \check{q} \cdot J_{\bar{y}(p)}\check{q}. \quad (172)$$

Since $J_{\bar{y}(p)}$ is always positive semidefinite, $m(\underline{\psi}) \geq 0$. We will now show that $m(\underline{\psi}) > 0$.

For any $(p, q) \in P \times \mathcal{S}^{L-2}$ for which $\check{q} \cdot J_s(p)\check{q} = 0$, and any $\tau \in \mathbb{R}$, we have, by the positive semidefiniteness of $J_{\bar{y}}(p)$,

$$0 \leq (\check{q} - \tau J_{\bar{y}}(p)\check{q}) \cdot J_{\bar{y}}(p)(\check{q} - \tau J_{\bar{y}}(p)\check{q}) \quad (173)$$

$$= \check{q} \cdot J_{\bar{y}}(p)\check{q} - 2\tau \check{q} \cdot J_{\bar{y}}(p)^2 \check{q} + \tau^2 \check{q} \cdot J_{\bar{y}}(p)^3 \check{q} \quad (174)$$

$$= -2\tau \|J_{\bar{y}}(p)\check{q}\|^2 + \tau^2 \check{q} \cdot J_{\bar{y}}(p)^3 \check{q}. \quad (175)$$

We cannot have $\check{q} \cdot J_{\bar{y}}(p)^3 \check{q} < 0$, or else (175) would be strictly negative for all $\tau > 0$. If $\check{q} \cdot J_{\bar{y}}(p)^3 \check{q} = 0$, it follows immediately that $\|J_{\bar{y}}(p)\check{q}\| = 0$. If $\check{q} \cdot J_{\bar{y}}(p)^3 \check{q} > 0$, observe that we must have

$$2\|J_{\bar{y}}(p)\check{q}\|^2 \leq \tau \check{q} \cdot J_{\bar{y}}(p)^3 \check{q} \quad \forall \tau > 0, \quad (176)$$

and thus that, again, $\|J_{\bar{y}}(p)\check{q}\| = 0$.

So if (p, q) is such that $\check{q} \cdot J_{\bar{y}}(p)\check{q} = 0$, then $J_{\bar{y}}(p)\check{q} = \mathbb{0}_{L-1}$. But then $\hat{\psi}(p, q) = \mathbb{0}_I$; so $(p, q) \in N(\underline{\psi})$; so $(p, q) \notin \mathcal{Q}(\underline{\psi})$. It follows that $m(\underline{\psi}) > 0$, as desired.

Since P is compact,

$$\bar{\bar{p}} \triangleq \max_{p \in P, \ell} p_\ell \quad (177)$$

is defined. Furthermore, it follows from $\bar{G}\bar{\delta} = 0$ that, for all ℓ , $\bar{G}e_\ell = \bar{G}\bar{b}_\ell$, where

$$\bar{b}_\ell \triangleq e_\ell - \bar{p}_\ell \bar{\delta} = e_\ell - \bar{p}_\ell e_L. \quad (178)$$

$\bar{b}_L = \mathbb{0}_L$. For $\ell < L$, e_ℓ and e_L are orthogonal, so $\|\bar{b}_\ell\| = \sqrt{1 + \bar{p}_\ell} \leq \sqrt{1 + \bar{\bar{p}}}$.

Note that $\bar{p} \cdot \bar{b}_\ell = 0 \quad \forall \ell$. Also, recall that $\bar{p} \in \bar{P}$ (not just $\in P$). Therefore $J_s(\bar{p})$ is defined, and by definition of \bar{G} ,

$$(J_s(\bar{p}) - J_{\chi_{[\bar{\psi}]}}(\bar{p}))\bar{G}\bar{b}_\ell = \bar{b}_\ell \quad (179)$$

$$\implies (\bar{G}\bar{b}_\ell) \cdot (J_s(\bar{p}) - J_{\chi_{[\bar{\psi}]}}(\bar{p}))\bar{G}\bar{b}_\ell = (\bar{G}\bar{b}_\ell) \cdot \bar{b}_\ell. \quad (180)$$

Since $J_{\hat{\chi}_{[\bar{\psi}]}}(\hat{\bar{p}})$ is negative definite, and the L^{th} entry of $\bar{G}\bar{b}_\ell$ equals zero, $(\bar{G}\bar{b}_\ell) \cdot J_{\chi_{[\psi]}}(\bar{G}\bar{b}_\ell) \leq 0$. So we have

$$(\bar{G}\bar{b}_\ell) \cdot J_s(\bar{p})\bar{G}\bar{b}_\ell \leq (\bar{G}\bar{b}_\ell) \cdot \bar{b}_\ell \quad (181)$$

$$\implies \|\bar{G}\bar{b}_\ell\| \left(\frac{\bar{G}\bar{b}_\ell}{\|\bar{G}\bar{b}_\ell\|} \right) \cdot J_s(\bar{p}) \left(\frac{\bar{G}\bar{b}_\ell}{\|\bar{G}\bar{b}_\ell\|} \right) \leq \frac{\bar{G}\bar{b}_\ell}{\|\bar{G}\bar{b}_\ell\|} \cdot \bar{b}_\ell \quad (182)$$

Because $\|\bar{G}\bar{b}_\ell/\|\bar{G}\bar{b}_\ell\|\| = 1$, $\|\bar{b}_\ell\| \leq \sqrt{1 + \bar{\bar{p}}}$, and the dot product of two vectors cannot exceed the product of their norms, the right-hand side cannot exceed

$\sqrt{1 + \bar{p}}$. Also, again because $\|\bar{G}\bar{b}_\ell / \|\bar{G}\bar{b}_\ell\|\| = 1$, the left-hand side must be at least $\|\bar{G}\bar{b}_\ell\|m(\underline{\psi})$. So we have

$$\|\bar{G}\bar{b}_\ell\|m(\underline{\psi}) \leq \sqrt{1 + \bar{p}} \quad (183)$$

$$\implies \|\bar{G}\bar{b}_\ell\| \leq \frac{\sqrt{1 + \bar{p}}}{m(\underline{\psi})}. \quad (184)$$

So, for any choice of $\underline{\psi}$, we know that, for each i and ℓ , either $|\psi_\ell^i(\bar{p}, \bar{\psi})| \leq \underline{\psi}$ or

$$|\psi_\ell^i(\bar{p}, \bar{\psi})| \leq \frac{\sqrt{1 + \bar{p}}}{m(\underline{\psi})} \max_{p \in P} [\nabla w^i(s(p)) \cdot J_{\bar{y}}(p) \mathbf{1}_L]. \quad (185)$$

As usual, we know the maximum exists by the continuity of the functions involved, the compactness of P , and the extreme value theorem.

Combining these bounds, we have

$$|\psi_\ell^i(\bar{p}, \bar{\psi})| \leq \bar{\psi}^i \triangleq \min_{\underline{\psi} \geq 0} \left(\max \left(\underline{\psi}, \frac{\sqrt{1 + \bar{p}}}{m(\underline{\psi})} \max_{p \in P} [\nabla w^i(s(p)) \cdot J_{\bar{y}}(p) \mathbf{1}_L] \right) \right). \quad (186)$$

Also, for L in particular, it follows immediately from $\bar{b}_L = \mathbf{0}_L$ that $\bar{G}e_L = \bar{G}\bar{b}_L = \mathbf{0}_L$, and thus that $\psi_L^i(\bar{p}, \bar{\psi}) = 0$.

So, with

$$\Psi \triangleq \{\psi \in \mathbb{R}^{L \times I} : |\psi_\ell^i| \leq \bar{\psi}^i \forall i, \ell < L; \psi_L^i = 0 \forall i\}, \quad (187)$$

we have $\psi(\bar{p}, \bar{\psi}) \in \Psi$.

Let Ψ characterize the $\{\bar{\psi}^i\}$ with respect to which the $\{t_\ell^i\}$ and $\{\omega_\ell^i\}$ satisfy conditions (73)–(75). Given $\psi \in \Psi$, let

$$p(\psi) \triangleq p \in \tilde{P} : \hat{z}_{[\psi]}(\hat{p}) = \mathbf{0}_L \quad (188)$$

denote the WES in \tilde{P} compatible with ψ . Recall that such a WES will exist, be unique up to rescaling (and thus unique in \tilde{P}), and be regular.

Because $s(\cdot)$ is \mathcal{C}^1 across \tilde{P} and $\chi_{[\cdot]}(\cdot)$ is \mathcal{C}^1 in both arguments (given $p \in \tilde{P}$), so is $z_{[\cdot]}(\cdot)$. So therefore is $\hat{z}_{[\cdot]}(\cdot)$, where, following (35),

$$\hat{z}_{[\psi]}(\hat{p}) \triangleq \mathcal{I} z_{[\psi]}(\dot{\hat{p}}). \quad (189)$$

Recall also that $J_{\hat{z}_{[\psi]}, \hat{p}}(\hat{p})$ is invertible for any (\hat{p}, ψ) for which \hat{p} is a regular normalized WES given ψ . So, by the IFT, $p(\cdot)$ is \mathcal{C}^1 .

We will now show that $G(p(\psi), \psi)$ is continuous in ψ throughout Ψ .

Suppose it is not. Then there exist $\bar{\psi}, \psi \in \Psi$ such that

$$\lim_{\epsilon \rightarrow 0} G(p(\bar{\psi} + \epsilon\psi), \bar{\psi} + \epsilon\psi) \neq G(p(\bar{\psi}), \bar{\psi}). \quad (190)$$

There must then be some $\Delta x^i \in \mathbb{R}^L$ such that

$$\lim_{\epsilon \rightarrow 0} G(p(\bar{\psi} + \epsilon\psi), \bar{\psi} + \epsilon\psi) \Delta x^i \neq G(p(\bar{\psi}), \bar{\psi}) \Delta x^i — \quad (191)$$

e.g., if the limit does not hold for some entry in column ℓ of G , $\Delta x^i = e_\ell$. This in turn implies

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} -J_{z_{[\bar{\psi} + \epsilon\psi]}}(p(\bar{\psi} + \epsilon\psi)) G(p(\bar{\psi} + \epsilon\psi), \bar{\psi} + \epsilon\psi) \Delta x^i &\neq \\ \lim_{\epsilon \rightarrow 0} -J_{z_{[\bar{\psi} + \epsilon\psi]}}(p(\bar{\psi} + \epsilon\psi)) G(p(\bar{\psi}), \bar{\psi}) \Delta x^i. \end{aligned} \quad (192)$$

But this is impossible: the left-hand side equals Δx^i for all ϵ , by definition of G , and the right-hand limit equals Δx^i because $p(\psi)$ is continuous and $-z_{[\psi]}(p)$ is \mathcal{C}^1 in p and ψ .

Defined on Ψ , $\psi(p(\psi), \psi)$ is thus a continuous function from a nonempty, compact, convex set to itself. By Brouwer's fixed point theorem, it has a fixed point. By construction, for any such fixed point ψ^* , $(p(\psi^*), \psi^*)$ is an RCESE.