Existential Risk and Exogenous Growth

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Abstract

Recent work has explored the relationship between economic growth and existential risk, using a model of population-driven endogenous growth and directed technical change. Within such a model, under moderate parameters, existential catastrophe is avoidable by a sufficiently rapid transition from consumption to safety production; but when the scale effect of existential risk is sufficiently large, existential catastrophe is inevitable. In a model with exogenous productivity growth, on the other hand, we find that existential catastrophe given a fixed population is not inevitable—even though productivity grows exogenously across the risk-increasing consumption sector and the risk-decreasing safety sector and is therefore not directed. We also find that existental catastrophe is avoidable regardless of the scale effect of existential risk.

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1 Introduction

Technological progress can bring both prosperity and danger. In particular, it can produce "existential risk": the risk of human extinction or a catastrophe that otherwise irreversibly curtails the potential of humankind, such as a war that permanently sends us back to the Stone Age (Bostrom (2002), Posner (2004), and Farquhar et al. (2017)).

Aschenbrenner (2020) develops a model of endogenous growth and directed technical change, involving a tradeoff between consumption and safety. The production of consumption goods carries some risk of disaster, which can be mitigated by spending on safety and developing safety technology. He finds that, in such a model, the probability of existential catastrophe depends critically on the scale effect of existential risk—that is, how proportionally growing both consumption and safety affects existential risk. If existential risk decreases with scale, no special concern for safety is required for the hazard rate to fall to zero exponentially. If existential risk increases with scale moderately, the existential hazard rate typically should follow an inverted U-shape. Finally, if the scale effect of existential risk is too large and the returns to research diminish too rapidly, it is impossible to avert an eventual existential catastrophe.

Note that much previous work on the economics of catastrophic risk (Martin and Pindyck (2015, 2019) and Aurland-Bredesen (2019)) had made the implicit assumption that the risk of catastrophe stays constant with scale. This is a knife-edge assumption: holding safety spending constant as a fraction of output only holds risk constant when the scale effect is exactly zero. Aschenbrenner generalizes from this assumption, illustrating the divergent dynamics of cases in which existential risk decreases, increases moderately, or increases rapidly with scale.

Because of his use of a semi-endogenous growth model, however, Aschenbrenner's results depend on a continued exponential (though, in principle, arbitrarily small) rate of population growth. This assumption stands in contrast to projections by the United Nations (2019) that the world population will plateau toward the end of this century less than 50% above its current level and remain at approximately this level for the foreseeable future—potentially even declining. In this scenario, as explored by Jones (2020), ideas-based growth models project stagnating or even regressing output.

I therefore explore a similar framework, retaining Aschenbrenner's risk model (including its potential for scale effects) but replacing the ideas-based endogenous growth model with a simple exogenous growth model in which productivity grows at a constant rate indepenent of population size. Productivity increases allow both consumption goods and safety goods to be produced with ever less labor. The assumption of exogenous growth follows other work on catastrophe mitigation, including that of Martin and Pindyck cited above and of the Nordhaus and Sztorc (2013) model of climate change abatement. Note, however, that Nordhaus models productivity growth as declining to zero over the coming centuries and models growth in output productivity and in emissions-reduction technology separately, whereas for simplicity, I assume that productivity growth is constant and identical across industries.

Like Aschenbrenner (and reminiscent of Jones (2016)), I find that as society grows wealthier, it will be optimal to spend an ever greater fraction of output on protection from existential risk. I also find that if existential risk decreases with scale, no special concern for safety is required for the hazard rate to fall to zero exponentially, and that if existential risk increases with scale, the existential hazard rate should typically follow an inverted U-shape. Whereas his results rely on exponential population growth, however, I find in the exogenous growth framework that existential catastrophe given a fixed population is not inevitable—even though productivity growth grows exogenously across the risk-increasing consumption sector and the risk-decreasing safety sector and is therefore not directed. I also find that a positive probability of avoiding existential catastrophe is achievable regardless of the scale effect of existential risk, but is not optimal to achieve if marginal utility in consumption diminishes sufficiently slowly.

The rest of this paper is organized as follows. Section 2 presents the economic environment of the model and a benchmark "rule of thumb allocation". Section 3 presents the asymptotic optimal growth path, highlighting how the scale effect of existential risk matters for the long run. Section 4 concludes.

2 The Economic Environment

2.1 Setup

The economy features a consumption sector, producing consumption good C_t, and a safety sector, producing safety good H_t. Total production in each sector is given by

$$C_t = A_t L_t \quad \text{and} \quad H_t = A_t S_t. \tag{1}$$

Total factor productivity A grows at some exogenous, constant exponential rate g. Note that, in this framework, raising A_t to some common exponent "α" in the C_t and H_t production functions is equivalent to the above where A_t grows at rate αg. Eliminating α is therefore without further loss of generality.

The size of the global labor force is normalized to 1 and assumed to be constant. Our resource constraint for labor is thus

$$L_t + S_t \leq 1. \tag{2}$$

An existential catastrophe results in permanent zero utility thereafter. Formally, human civilization faces a time-varying hazard rate δ_t, representing a stochastic probability of existential catastrophe. The probability that human civilization survives to date t (starting from date 0) is given by

$$M_t = e^{-\int_0^t \delta_s ds}, \tag{3}$$

corresponding to the laws of motion

$$\dot{M}_t = -\delta_t M_t, \quad M_0 = 1. \tag{4}$$

The hazard rate is endogenous, and as explained above increases with consumption and decreases with safety spending:

$$\delta_t = \overline{\delta} C_t^{\epsilon} H_t^{-\beta}. \tag{5}$$

Expected lifetime utility for a representative agent is

$$U = \int_0^\infty e^{-\rho t} u(C_t) M_t dt, \tag{6}$$

where flow utility is isoelastic in consumption:

$$u(C_t) = \overline{u} + \frac{C_t^{1-\gamma}}{1-\gamma}. \tag{7}$$

The parameter $\overline{u}$ is a constant that specifies the upper bound of the utility of life relative to death (with the utility of death implicitly normalized to 0) in the case where $\gamma > 1$ and thus $c^{1-\gamma}/(1-\gamma)$ is negative.

Flow utility is discounted at exponential rate ρ, representing the (positive) sum of some nonnegative rate of pure time preference and some nonnegative rate of natural and unavoidable existential risk.

$M_\infty = \lim_{t\to\infty} M_t = e^{-\int_0^\infty \delta_s ds}$ represents the probability that human civilization does *not* succumb to an anthropogenic existential catastrophe and, at least in expectation, enjoys a long and flourishing future.[1] Note that $M_\infty > 0$ iff $\int_0^\infty \delta_s ds$ is bounded.

The representative agent, or his social planner, faces a single allocative decision: the fraction of workers L_t to allocate to consumption (vs. safety) production at all times t.

2.2 Rule of Thumb Allocation

As a benchmark, we will consider a simple "rule of thumb" allocation, as in Jones (2016) and Aschenbrenner (2020). This rule of thumb allocation is analogous to Solow's (1956) assumption of a fixed saving rate in his version of the neoclassical growth model. In particular, we will consider an allocation in which the fraction of labor working on consumption is fixed over time. Later, we will consider the optimal allocation, in which the fraction of resources dedicated to safety can evolve.

Throughout the results that follow, we will use the notation $g_x^* \equiv \lim_{t\to\infty} g_{xt}$, where $g_{xt} \equiv \dot{x}_t/x_t$, for each time-dependent variable x.

Proposition 1. ***Balanced growth under rule of thumb allocation*** *Consider a rule of thumb allocation in which $L_t = L^* \in (0,1)$. There exists a balanced growth path in which the growth rates of consumption output, safety output, and the hazard rate are given respectively by*

$$g_{Ct} = g_C^* = g_{Ht} = g_H^* = g, \tag{8}$$

$$g_{\delta t} = g_\delta^* = g(\epsilon - \beta). \tag{9}$$

with

$$\delta_t = \delta^* = \overline{\delta}\left(\frac{L^*}{1-L^*}\right)^{\epsilon} > 0 \text{ if } \epsilon = \beta. \tag{10}$$

[1] In the face of natural existential risk, this will entail eventually succumbing to a natural existential catastrophe instead. From very-long-run historical data on large-scale natural catastrophes, however, Snyder-Beattie et al. (2019) estimate the annual natural existential hazard rate to be no more than 1 in 14,000, and likely much lower.

Proof. See Appendix A.1. □

As we can see, $\delta_t \to 0$ if $\epsilon < \beta$ and $\delta_t \to \infty$ if $\epsilon > \beta$.

What is important to note about this rule of thumb allocation is what happens to existential risk. If $\epsilon < \beta$, i.e. if proportional increases to safety spending reduce the hazard rate δ by more than proportional increases to consumption increase it, δ falls to zero at an exponential rate. Therefore, $\int_0^\infty \delta_s ds$ is bounded, which implies that the long-run probability of human civilization's survival, M_∞, is strictly greater than zero. In the knife-edge case of $\epsilon = \beta$, the hazard rate always equals a positive constant, implying $M_\infty = 0$. If $\epsilon > \beta$, the hazard rate increases exponentially. This induces not only $M_\infty = 0$, but in fact $\delta \to \infty$; that is, the instantaneous probability of an existential catastrophe approaches 1.

3 The Optimal Allocation

Now consider a representative agent that maximizes its utility. The representative agent discounts future utility with positive rate ρ, due to impatience and/or to exogenous existential risk, as noted above.

The optimal allocation of resources is a time path for L_t, M_t, δ_t that maximizes the utility of the representative agent, solving the following problem:

$$\max_{\{L_t\}} U = \int_0^\infty M_t u(C_t) e^{-\rho t} dt, \tag{11}$$

subject to

$$C_t = A_t L_t, \tag{12}$$
$$H_t = A_t S_t, \tag{13}$$
$$L_t + S_t = 1, \tag{14}$$
$$\dot{A}_t = A_t g, \tag{15}$$
$$\dot{M}_t = -\delta_t M_t, \tag{16}$$
$$\delta_t = \bar{\delta} C_t^{\epsilon} H_t^{-\beta}. \tag{17}$$

To solve for the optimal allocation, I define the current value Hamiltonian:

$$\mathcal{H} = M_t u(C_t) - v_t \delta_t M_t, \tag{18}$$

where L_t is our single control variable and M_t is our single state variable. The costate variable v_t captures the shadow values of an extra "lifetime". Based on the maximum principle and the arguments of Romer (1986), the first-order conditions characterize a solution. [TODO: Verify that this applies here.]

It will be useful to define

$$\tilde{v}_t \equiv \frac{v_t}{u'(C_t)\, C_t}. \tag{19}$$

This is the shadow value of life, converted to consumption units by $u'(C_t)$, as a ratio to the level of consumption.

After some manipulation (see Appendix A.2) the first order conditions yield:

$$\frac{S_t}{L_t} = \frac{\beta \delta_t \tilde{v}_t}{1 - \epsilon \delta_t \tilde{v}_t}, \tag{20}$$

$$\rho = \frac{\dot{v}_t}{v_t} + \frac{1}{v_t}(u(C_t) - v_t \delta_t). \tag{21}$$

The term $\tilde{v}_t$—and in particular the product $\delta_t \tilde{v}_t$—thus determines the allocation of workers to consumption vs. safety. From (21), v_t can also be represented as

$$v_t = \frac{u(C_t)}{\rho + \delta_t - g_{vt}}, \tag{22}$$

and thus

$$\tilde{v}_t = \frac{\tilde{u}_t}{\rho + \delta_t - g_{vt}}, \quad \tilde{u}_t = \frac{u(C_t)}{u'(C_t)C_t}. \tag{23}$$

$\tilde{u}_t$ is the opportunity cost of death $u(C_t)$, converted into consumption units by $u'(C_t)$, divided by the level of consumption C_t. $\tilde{u}$ thus represents the relative value of life. The denominator of $\tilde{v}_t$ essentially converts this into a discounted present value. Therefore, $\tilde{v}_t$ represents the discounted relative value of life and determines the demand for safety.

Note that the allocation of labor to safety is proportional to $\frac{\beta \delta_t \tilde{v}_t}{1 - \epsilon \delta_t \tilde{v}_t}$. The numerator represents the marginal value of work on safety: the reduction in the hazard rate (times the discounted relative value of life). The denominator represents the marginal value of consumption: the utility benefits of consumption, normalized to 1, minus the increase in the hazard rate (times the discounted relative value of life). As in Aschenbrenner (2020) and unlike in Jones (2016), $\delta_t \tilde{v}_t$ cannot rise forever, since if $\epsilon \delta_t \tilde{v}_t > 1$ the marginal value of consumption is negative.

3.1 The Optimal Allocation with $\epsilon \leq \beta$

First, consider the case in which safety goods are at least as potent in reducing existential risk as consumption goods in increasing existential risk, i.e. $\epsilon \leq \beta$. Then, existential risk weakly decreases with scale. The asymptotic growth path depends on the curvature of our preferences.

Proposition 2. ***Optimal growth with*** $\epsilon \leq \beta$ ***and*** $\gamma > 1 + \beta - \epsilon$
Assume that $\epsilon \leq \beta$ *and that the marginal utility of consumption falls rapidly, in the sense that* $\gamma > 1 + \beta - \epsilon$*. Then the optimal allocation features an asymptotic constant growth path such that as* $t \to \infty$*, asymptotic growth is given by:*

$$g_C^* = g\frac{\beta}{\gamma - 1 + \epsilon} > 0, \tag{24}$$

$$g_L^* = -g\Big(1 - \frac{\beta}{\gamma - 1 + \epsilon}\Big) < 0, \tag{25}$$

$$g_H^* = g > 0, \tag{26}$$

$$g_S^* = 0, \tag{27}$$

$$g_\delta^* = -g\Big(1 + \frac{\beta(\gamma - 1)}{\gamma - 1 + \epsilon}\Big) < 0. \tag{28}$$

Proof. See Appendix A.3. □

Note that $\delta_t \to 0$ exponentially, implying $M_\infty > 0$. Finally, note that this solution is valid for all $\rho > 0$.

Unlike in the rule of thumb allocation, the allocation of resources to safety can adjust. In particular,

$$\tilde{u}_t = \frac{u(C_t)}{u'(C_t)C_t} = \overline{u}C_t^{\gamma-1} + \frac{1}{1-\gamma}. \tag{29}$$

Thus, given $\gamma > 1$, the relative value of life $\tilde{u}_t$ increases as consumption grows. As people grow wealthier, the marginal utility of consumption declines, and it becomes relatively more valuable to spend on avoiding existential catastrophe. This happens regardless of discount rate ρ: no particular concern for the future is necessary for this dynamic. The rising value of life means that resources are shifted towards the safety sector. Consumption growth is substantially less than what is feasible and substantially less than safety growth.

Proposition 3. ***Optimal growth with*** $\epsilon \leq \beta$ ***and*** $\gamma \leq 1 + \beta - \epsilon$
Assume that $\epsilon \leq \beta$ *and that the marginal utility of consumption falls slowly, in the sense that* $\gamma \leq 1 + \beta - \epsilon$. *Then the optimal allocation features an asymptotic constant growth path such that as* $t \to \infty$, *asymptotic growth is given by:*

$$g_C^* = g > 0, \tag{30}$$
$$g_L^* = 0; \tag{31}$$

if $\gamma > 1$,

$$g_H^* = g\frac{\gamma + \epsilon}{1 + \beta} > 0, \tag{32}$$
$$g_S^* = -g\frac{1 + \beta - \epsilon - \gamma}{1 + \beta} \leq 0, \tag{33}$$
$$g_\delta^* = -g\frac{\beta\gamma - \epsilon}{1 + \beta} < 0; \tag{34}$$

if $\gamma \leq 1$,

$$g_H^* = g\frac{1 + \epsilon}{1 + \beta} > 0, \tag{35}$$
$$g_S^* = g\frac{\epsilon - \beta}{1 + \beta} \leq 0, \tag{36}$$
$$g_\delta^* = -g\frac{\beta(2 + \beta) - \epsilon}{1 + \beta} < 0. \tag{37}$$

In the edge cases that $\epsilon < \beta$ *and* $\gamma = 1 + \beta - \epsilon$, *or* $\epsilon = \beta$ *and* $\gamma \leq 1$, *we have* $g_H^* = g$ *and* $g_S^* = 0$. S_t *and* L_t *approach constants strictly between zero and one. Otherwise,* $g_S^* < 0$ *and so* $S_t \to 0$.

Proof. See Appendix A.4. □

Note again that $\delta_t \to 0$ exponentially in all cases, implying $M_\infty > 0$.

In short, when γ is smaller than $1 + \beta - \epsilon$, the value of life does not grow faster than the hazard rate δ_t declines. Thus, the critical product $\delta_t \tilde{v}_t$ declines, and resources are shifted to consumption. Consumption growth is, in the limit, as fast as is feasible. At the same time, the hazard rate falls to 0.

In this sense, the outcome of the optimal allocation is broadly similar to that of the rule of thumb allocation when $\epsilon < \beta$. It is possible to improve upon the rule of thumb allocation by shifting the allocation over time, but humanity has positive probability of survival in any case.

3.2 The Optimal Allocation with $\epsilon > \beta$

Now consider the case where consumption goods are more potent in increasing existential risk than safety goods are in reducing it, i.e. $\epsilon > \beta$.

Proposition 4. ***Optimal growth with*** $\epsilon > \beta$ ***and*** $\gamma > 1$
Assume that $\epsilon > \beta$ and that the marginal utility of consumption falls rapidly, in the sense that $\gamma > 1$. Then the optimal allocation features an asymptotic constant growth path such that as $t \to \infty$, asymptotic growth is given by:

$$g_C^* = g \frac{\beta}{\gamma - 1 + \epsilon} > 0, \tag{38}$$

$$g_L^* = -g\Big(1 - \frac{\beta}{\gamma - 1 + \epsilon}\Big) < 0, \tag{39}$$

$$g_H^* = g > 0, \tag{40}$$

$$g_S^* = 0, \tag{41}$$

$$g_\delta^* = -g\Big(1 + \frac{\beta(\gamma - 1)}{\gamma - 1 + \epsilon}\Big) < 0. \tag{42}$$

Proof. See Appendix A.5. □

Given $\gamma > 1$, the relative value of life $\tilde{u}_t$ rises as consumption grows, as when $\epsilon \leq \beta$. Now, however, we have $\epsilon - \beta > 0$: existential risk grows with scale. Despite this scale effect, workers and scientists are shifted to the safety sector quickly enough that δ_t still declines exponentially on the asymptotic growth path, yielding $M_\infty > 0$. Unlike in the rule of thumb allocation, there is a positive probability that humanity does succumb to an existential catastrophe.

Proposition 5. ***Optimal growth with*** $\epsilon > \beta$ ***and*** $1 - \epsilon < \gamma \leq 1$
Assume that $\epsilon > \beta$ and that the marginal utility of consumption falls slowly, in the sense that $\gamma \leq 1$. Finally, assume that the elasticity of the hazard rate is larger than the elasticity of utility with respect to consumption, i.e. $\epsilon > 1-\gamma$. Then, the optimal allocation features an asymptotic constant growth path such that as $t \to \infty$, asymptotic growth is given by:

$$g_C^* = \frac{\beta}{\epsilon} g > 0, \tag{43}$$

$$g_L^* = \frac{\beta - \epsilon}{\epsilon} g < 0, \tag{44}$$

$$g_H^* = g > 0, \tag{45}$$

$$g_S^* = 0, S_t \to 1, \tag{46}$$

$$g_\delta^* = 0, \delta_t \to \delta^* = \frac{(1-\gamma)\rho - (1-\gamma)^2 g_C^*}{\epsilon - 1 + \gamma}. \tag{47}$$

Proof. See Appendix A.6. □

Given $\gamma \leq 1$, the relative value of life $\tilde{u}_t$ is bounded, just as when $\epsilon < \beta$. Now, however, we have $\epsilon - \beta > 0$: existential risk grows with scale. Unlike when $\epsilon < \beta$, labor is shifted to the safety sector even when $\gamma \leq 1$, not just the narrower class of preferences with γ significantly greater than one as in Jones (2016). This is because even though the relative value of life $\tilde{u}_t$ is bounded when $\gamma \leq 1$, δ_t continues to increase because of the scale effect of existential risk. When $\epsilon > 1 - \gamma$, $\delta_t \tilde{v}_t$ would get too large if labor wasn't shifted to safety. Nevertheless, despite labor being shifted to safety, it is not shifted quickly enough to bound $\int_0^\infty \delta_s ds$, so the long-run probability of humanity's survival is $M_\infty = 0$ when $\gamma \leq 1$ and $\epsilon > \beta$.

However, there is also another case when $\epsilon < 1 - \gamma$. Here, the elasticity of the hazard rate is smaller than the elasticity of utility with respect to consumption, so resources need not be all shifted to safety to bound $\delta_t \tilde{v}_t$, and so the optimal allocation features balanced growth as in the rule of thumb allocation and $\delta_t \to \infty$.

Proposition 6. ***Optimal growth with*** $\epsilon > \beta$ ***and*** $\gamma < 1 - \epsilon$
Assume that $\epsilon > \beta$ and that the marginal utility of consumption falls slowly, in the sense that $\gamma < 1 - \epsilon$. Then, the optimal allocation features an asymptotic constant growth path such that as $t \to \infty$, asymptotic growth is given

by:

$$g_C^* = g > 0, \tag{48}$$

$$g_L^* = 0, L_t \to L^* = \frac{1-\gamma-\epsilon}{\beta+1-\gamma-\epsilon}, \tag{49}$$

$$g_H^* = g > 0, \tag{50}$$

$$g_S^* = 0, S_t \to S^* = \frac{1-\gamma-\epsilon}{\beta+1-\gamma-\epsilon}, \tag{51}$$

$$g_\delta^* = (\epsilon-\beta)g > 0. \tag{52}$$

Proof. See Appendix A.7. □

Critically, consider the comparison of the optimal allocation to the rule of thumb allocation. In the rule of thumb allocation when $\epsilon > \beta$, $\delta_t \to \infty$ and $M_\infty = 0$ because of the scale effect of existential risk. By contrast, when $\gamma > 1-\epsilon$, resources are shifted to the safety sector in optimal allocation, counteracting the scale effect. Thus, δ_t converges to a small constant or even zero. Given $\gamma > 1$, the optimal allocation features δ_t falling to zero exponentially, and thus $M_\infty > 0$. However, if $\gamma \leq 1$, $M_\infty = 0$. Nonetheless, if the marginal utility of consumption falls sufficiently slowly, i.e. $\gamma < 1-\epsilon$, not all labor is shifted to the safety sector asymptotically and the optimal allocation looks like the rule of thumb allocation, with the hazard rate growing exponentially.

The case of $\epsilon > \beta$ is thus a world in which existential risk is an enormous challenge, but can still be overcome. With a static concern for safety, as in the rule of thumb allocation, the scale effect portends disaster. By shifting resources to safety, as in the optimal allocation for sufficiently curved preferences, this scale effect can be contained; in fact, when $\gamma > 1$, even the impatient optimal allocation features a nonzero probability of humanity's survival in the long run.

3.3 Summary

To provide an overview of the various optimal allocations, here is an overview of the asymptotic growth paths under different parameter values. For the sake of clarity, I have omitted the knife-edge cases.

Table 1: Overview of Optimal Allocations

<table>
<tr><th></th><th>$\epsilon < \beta$</th><th>$\epsilon > \beta$</th></tr>
<tr><td>Rule of thumb allocation</td><td>$L_t = L^* \in (0,1)$
$g_C = g_H = g$
$\delta_t \to 0$
$M_\infty > 0$</td><td>$L_t = L^* \in (0,1)$
$g_C = g_H = g$
$\delta_t \to \infty$
$M_\infty = 0$</td></tr>
<tr><td>Optimal allocation with smallest γ</td><td rowspan="2">$L_t \to 1$
$g_C \to g$
$\delta_t \to 0$
$M_\infty > 0$</td><td>$L_t \to L^* \in (0,1)$
$g_C \to g$
$\delta_t \to \infty$
$M_\infty = 0$</td></tr>
<tr><td>Optimal allocation with smaller γ</td><td>$L_t \to 0$
$g_C \to g_C^* \in (0,g)$
$\delta_t \to \delta^* > 0$
$M_\infty = 0$</td></tr>
<tr><td>Optimal allocation with large γ</td><td>$L_t \to 0$
$g_C \to g_C^* \in (0,g)$
$\delta_t \to 0$
$M_\infty > 0$</td><td>$L_t \to 0$
$g_C \to g_C^* \in (0,g)$
$\delta_t \to 0$
$M_\infty > 0$</td></tr>
<tr><td>Existential risk in rule of thumb vs. optimal allocation</td><td>δ exponentially decays under rule of thumb. Optimal allocation changes pace of decay.</td><td>δ explodes under rule of thumb. Optimal allocation can contain growth in δ.</td></tr>
</table>

4 Conclusion

Aschenbrenner (2020) explores the possible relationship between economic growth and existential risk, using an endogenous growth model in which sustained output growth requires sustained population growth, and an existental risk model in which the production of consumption goods increases existential risk and the production of "safety goods" lowers it. He finds that, as society grows richer, it should allocate ever more of its resources from consumption to existential risk mitigation efforts. Furthermore, he finds that this policy can render (at least anthropogenic) existential catastrophe permanently avoidable under circumstances in which it would be inevitable under a "rule of thumb" policy in which a fixed proportion of output is spent on risk reduction. Finally, he finds that if there is a sufficiently large "scale effect" of existential risk, existential catastrophe is unavoidable: lowering the hazard rate steeply enough to render the probability of long-run survival positive would require a consumption sacrifice so large that life would no longer be worth living, and which would therefore in itself constitute a kind of existential catastrophe.

Here, we have seen that the last of these findings, but not the rest, disappears when we use the same risk model but move to a simple exogenous growth model. The intuition for this divergence is straightforward. Fixing consmption at some positive-utility level, exponential exogenous productivity growth allows for an exponential increase in risk-mitigation efforts, which delivers a positive probability of survival. Exponential population growth with fixed consumption per capita, on the other hand, creates exponential increases in output only at the expense of corresponding exponential increases in the risky production of consumption goods.

Since we are unlikely to sustain population growth in coming centuries, but may nonetheless sustain productivity growth, the above is good news for those concerned with humanity's long-term survival. It also illustrates the potential importance of growth theory from a longtermist perspective, even if we believe with Bostrom (2003) that existential risk is the primary determinant of the expected value of the future. Our beliefs about the feasibility of containing existential risk may be sensitive to our beliefs about the mechanisms driving economic growth.

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Appendix

A.1 Proof of Proposition 1

By assumption, $C_t = A_t L_t$ and $H_t = A_t S_t$. Since L_t and S_t are constant in this rule of thumb allocation, C_t and H_t grow at the growth rate of A, denoted g. That is, $g_{Ct} = g_{Ht} = g\ \forall t$. So

$$g_{Ct} = g_{H,t} = g_C^* = g_H^* = g\ \forall t. \tag{53}$$

Since $\delta_t = \overline{\delta} C_t^{\epsilon} H_t^{-\beta}$, we have $g_{\delta t} = \epsilon g_{Ct} - \beta g_{Ht}$. From the above, $g_{\delta t} = \epsilon g - \beta g\ \forall t$. So

$$g_{\delta t} = g_\delta^* = g(\epsilon - \beta)\ \forall t. \tag{54}$$

Finally, if $\epsilon = \beta$, $\delta_t = \overline{\delta}(C_0 e^{g_C^* t})^{\epsilon}(H_0 e^{g_H^* t})^{-\epsilon} = \overline{\delta}(C_0/H_0)^{\epsilon}\ \forall t$. Since $C_0 = A_0 L^*$ and $H_0 = A_0(1 - L^*)$, we have $C_0/H_0 = L^*/(1 - L^*)$, and thus

$$\delta_t = \delta^* = \overline{\delta}\Big(\frac{L^*}{1 - L^*}\Big)^{\epsilon} > 0\ \forall t. \tag{55}$$

A.2 First Order Conditions of the Hamiltonian

FOC: L_t

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial L_t} &= 0 \\
\implies \frac{\partial}{\partial L_t}[M_t u(C_t) - v_t \delta_t M_t] &= 0 \\
\implies M_t \frac{\partial}{\partial L_t}\Big[\frac{(A_t L_t)^{1-\gamma}}{1-\gamma} + \overline{u} - v_t \overline{\delta} A_t^{\epsilon-\beta} L_t^{\epsilon} S_t^{-\beta}\Big] &= 0 \\
\implies v_t \overline{\delta} A_t^{\epsilon-\beta}(\epsilon L_t^{\epsilon-1}(1-L_t)^{-\beta} + L_t^{\epsilon}\beta(1-L_t)^{-\beta-1}) &= A_t^{1-\gamma} L_t^{-\gamma} \qquad (56)
\end{aligned}$$

Define

$$\tilde{v}_t \equiv \frac{v_t}{u'(C_t)C_t}. \qquad (57)$$

From $\tilde{v}_t = v_t C_t^{\gamma-1} = v_t(A_t L_t)^{\gamma-1}$, we have $A_t^{1-\gamma} L_t^{-\gamma} = v_t / L_t \tilde{v}_t$. It follows that

$$\begin{aligned}
\overline{\delta} A_t^{\epsilon-\beta} L_t^{\epsilon}(1-L_t)^{-\beta} \cdot \frac{\epsilon}{L_t} + \overline{\delta} A_t^{\epsilon-\beta} L_t^{\epsilon}(1-L_t)^{-\beta} \cdot \frac{\beta}{1-L_t} &= \frac{1}{L_t \tilde{v}_t} \qquad (58) \\
\implies \delta_t \Big(\frac{\epsilon}{L_t} + \frac{\beta}{1-L_t}\Big) &= \frac{1}{L_t \tilde{v}_t} \qquad (59) \\
\implies \delta_t \epsilon + \delta_t \beta \frac{L_t}{1-L_t} &= \frac{1}{\tilde{v}_t} \qquad (60) \\
\implies \frac{S_t}{L_t} &= \frac{\beta \delta_t \tilde{v}_t}{1 - \epsilon \delta_t \tilde{v}_t}. \qquad (61)
\end{aligned}$$

In other words, the ratio of worker-types is proportional to the ratio of the value of what these worker-types can produce. In the numerator is the hazard rate times the relative value of life times β (the effectiveness of safety goods in reducing existential risk), which is the marginal value of safety output. In the denominator is 1 (which is value of consumption relative to $\tilde{v}_t$) minus the risk-increasing effects of consumption, which is the marginal net value of consumption output.

FOC: M_t

$$\frac{\partial \mathcal{H}/\partial M_t + \dot{v}_t}{v_t} = \rho \tag{62}$$

$$\implies \frac{\dot{v}_t}{v_t} + \frac{1}{v_t}(u(C_t) - v_t\delta_t) = \rho \tag{63}$$

Transversality Condition

Note that the standard transversality condition applies:

$$\lim_{t\to\infty}[e^{-\rho t} \cdot v_t M_t] = 0 \tag{64}$$

The path of $\delta_t\tilde{v}_t$

Note that we must have $0 < L_t \leq 1$, where the strict inequality comes from the fact that at least some labor must be allocated to consumption along the balanced growth path. Thus S_t/L_t must be finite; its denominator cannot be 0. Given that ϵ, β, δ_t, and $\tilde{v}_t$ are guaranteed to be positive along the optimal path, we have

$$\delta_t\tilde{v}_t < \frac{1}{\epsilon}. \tag{65}$$

This foreshadows what will happen along the balanced growth path. Given the preference parameter, either δ_t falls to 0 faster than $\tilde{v}_t$, in which case $\delta_t\tilde{v}_t$ falls to 0, or $\delta_t\tilde{v}_t$ asymptotically approaches $1/\epsilon$.

Characterizing $\tilde{v}_t$

From FOC: M_t, we obtain

$$\frac{\dot{v}_t}{v_t} + \frac{1}{v_t}(u(C_t) - v_t\delta_t) = \rho$$

$$\implies \rho - \frac{\dot{v}_t}{v_t} + \delta_t = \frac{u(C_t)}{v_t}$$

$$\implies v_t = \frac{u(C_t)}{\rho + \delta_t - g_{vt}} \tag{66}$$

$$\implies \tilde{v}_t = \frac{u(C_t)/u'(C_t)C_t}{\rho + \delta_t - g_{vt}}. \tag{67}$$

Thus, the relative value of life depends on the extra utility a person enjoys versus increasing consumption on the current margin. This is why the degree of diminishing returns, γ, in our utility function plays such a key role.

Finally, given isoelastic utility,

$$\begin{aligned} \frac{u(C_t)}{u'(C_t)C_t} &= \frac{\overline{u} + \frac{C_t^{1-\gamma}}{1-\gamma}}{C_t^{-\gamma} C_t} \\ \implies \frac{u(C_t)}{u'(C_t)C_t} &= (\overline{u} + \frac{C_t^{1-\gamma}}{1-\gamma})(C_t^{-(1-\gamma)}) \\ \implies \frac{u(C_t)}{u'(C_t)C_t} &= \overline{u}C_t^{\gamma-1} + \frac{1}{1-\gamma}. \end{aligned} \tag{68}$$

A.3 Proof of Proposition 2

It follows from $\epsilon \leq \beta$ and $\gamma > 1 + \beta - \epsilon$ that $\gamma > 1$. Observe that when $\gamma > 1$, given equations (68) and (67), along a balanced growth path in which $C_t \to \infty$:

$$\begin{aligned} g_{\tilde{v}} &= g_{\frac{u(C_t)}{u'(C_t)C_t}} - g_{\rho+\delta_t-g_{vt}} \\ &= g_{\overline{u}C_t^{\gamma-1}+\frac{1}{1-\gamma}} \\ &= (\gamma-1)g_C \end{aligned} \tag{69}$$

as long as δ_t converges to some constant.

Now let us conjecture that when $\epsilon \geq \beta$ and $\gamma > 1 + \beta - \epsilon$, there is an asymptotic balanced growth path in which C_t and H_t rise, and L_t falls to zero, at constant exponential rates.

Since $C_t = A_t L_t$ and $H_t = A_t S_t$, in our proposed solution consumption growth and safety growth are given by

$$\begin{aligned} g_C &= g + g_L, \\ g_H &= g + g_S. \end{aligned} \tag{70}$$

Substituting the g_C term into (95):

$$g_{\tilde{v}} = (\gamma-1)(g+g_L). \tag{71}$$

Now recall from (61) that $(1-L_t)/L_t = (\beta\delta_t\tilde{v}_t)/(1-\epsilon\delta_t\tilde{v}_t)$. Given a constant, positive ϵ and β, the only way for L_t to fall to 0 is for $\delta_t\tilde{v}_t$ to grow. From (65), however, $\delta_t\tilde{v}_t < 1/\epsilon$. Thus, as $t \to \infty$, $\delta_t\tilde{v}_t \to 1/\epsilon$, i.e. $\delta_t\tilde{v}_t$ is asymptotically constant. This in turn means that $\epsilon\delta_t\tilde{v}_t$ converges to 1 asymptotically, meaning that $1-\epsilon\delta_t\tilde{v}_t$ will fall to 0 exponentially. This then delivers the desired exponential increase in $(1-L_t)/L_t$ and the exponential fall to 0 of L_t.

We now have:

$$\begin{aligned} &\lim_{t\to\infty} \dot{\ln}(\delta_t\tilde{v}_t) = 0 \\ \implies\ & g_\delta = -g_{\tilde{v}}. \end{aligned} \tag{72}$$

From (71):

$$g_\delta = -(\gamma-1)(g+g_L) \tag{73}$$

From $\delta_t = \overline{\delta} A_t^{\epsilon-\beta} L_t^{\epsilon}(1-L_t)^{-\beta}$, we can produce another expression for g_δ:

$$\begin{aligned} g_\delta &= \lim_{t\to\infty} \dot{\ln}(\overline{\delta} A_t^{\epsilon-\beta} L_t^{\epsilon}(1-L_t)^{-\beta}) \\ &= \lim_{t\to\infty} \Big[(\epsilon-\beta)\dot{\ln}(A_t) + \epsilon\dot{\ln}(L_t) - \beta\dot{\ln}(1-L_t)\Big] \\ &= (\epsilon-\beta)g + \epsilon g_L \end{aligned} \tag{74}$$

where $\dot{\ln}(1-L_t) = g_S = 0$, because $1-L_t$ is asymptotically constant at 1.

Setting (74) equal to (73):

$$\begin{aligned} (\epsilon-\beta)g + \epsilon g_L &= -(\gamma-1)(g+g_L) \\ \implies ((\epsilon-\beta)+(\gamma-1))g &= (1-\gamma-\epsilon)g_L \\ \implies g_L &= g\frac{\epsilon-\beta+\gamma-1}{1-\gamma-\epsilon} \\ &= -g\Big(1-\frac{\beta}{\gamma-1+\epsilon}\Big). \end{aligned} \tag{75}$$

Given $\gamma > 1$, we have $g_L < 0$ iff

$$\begin{aligned} &\epsilon-\beta+\gamma-1 > 0 \\ \iff &\gamma > 1+\beta-\epsilon, \end{aligned} \tag{76}$$

as conjectured.

We can now calculate the other asymptotic growth rates as well. From (70) and (75),

$$g_C = g\frac{\beta}{\gamma-1+\epsilon}, \tag{77}$$

$$g_H = g. \tag{78}$$

$g_C > 0$ follows from $\gamma > 1$.

Finally, from (74) and (75),

$$\begin{aligned} g_\delta &= g(\epsilon-\beta) - \epsilon g(1-\frac{\beta}{\gamma-1+\epsilon}) \\ &= -g\Big(1+\frac{\beta(\gamma-1)}{\gamma-1+\epsilon}\Big). \end{aligned} \tag{79}$$

$g_\delta < 0$, and thus $\delta_t \to 0$, follow from $\gamma > 1$.

How Low Can ρ Go?

What values of ρ are permissible for our asymptotic growth path to be valid?

In particular, the denominator of our shadow price must be positive and the optimal allocation must satisfy the transversality condition.

First, consider the shadow price, v_t. Recall that

$$v_t = \frac{u(C_t)}{\rho + \delta - g_{vt}} \tag{80}$$

Along a balanced growth path, $g_{vt} \to g_v$, and we require that the denominator is positive. If $\delta \to \infty$, the denominator is clearly positive on a balanced growth path. If $\delta \to \delta^* \geq 0$, the denominator will be asymptotically constant and:

$$\begin{aligned} g_{vt} = g_u - g_{\rho+\delta-g_{vt}} = g_u &= (1-\gamma)g_C, & \gamma < 1; \\ &= 0, & \gamma \geq 1. \end{aligned} \tag{81}$$

Therefore, for the denominator of v_t to be positive, we require that $\rho+\delta > (1-\gamma)g_C^*$. This means that if $\delta \to 0$ and $\gamma \geq 1$ any $\rho > 0$ is valid. If $\delta \to 0$ and $\gamma < 1$, $\rho > (1-\gamma)g_C^*$ is valid. If $\delta \to \delta^* > 0$, $\rho > (1-\gamma)g_C^* - \delta$ is valid (with the condition that $\rho > 0$ still).

Now consider the transversality condition:

$$\lim_{t\to\infty}[e^{-\rho t} \cdot v_t M_t] = 0 \tag{82}$$

Note that whatever the behaviour of δ_t, $g_M^* = 0$. Considering the growth rate of the expression inside the limit, we require that:

$$-\rho + g_v + g_M^* < 0 \Rightarrow \rho > g_v \tag{83}$$

If $\delta \to \delta^* \geq 0$, this condition is just that $\rho > (1-\gamma)g_C^*$ as before. If $\delta \to \infty$ (as in Proposition 6), $g_v = g_{\tilde{v}} + (1-\gamma)g_C^* = (1-\gamma-\epsilon+\beta)g$. Then, for the transversality condition to be satisfied, $\rho > (1-\gamma-\epsilon+\beta)g$. Note that this is stricter than the requirement that $\rho > 0$, as $1-\gamma-\epsilon > 0$ in Proposition 6 and $\beta > 0$.

A.4 Proof of Proposition 3

Let us conjecture that when $\epsilon \leq \beta$ and $\gamma > 1 + \beta - \epsilon$, there is an asymptotic balanced growth path such that C_t and H_t rise, and S_t falls to zero, at constant exponential rates.

As in (70), in our proposed solution consumption growth and safety growth are given by

$$\begin{aligned} g_C &= g + g_L, \\ g_H &= g + g_S. \end{aligned} \tag{84}$$

Here, however, $L_t \to 1$, so we have $g_L = 0$ (rather than $g_S = 0$).

From $\delta_t = \overline{\delta} A_t^{\epsilon-\beta} L_t^{\epsilon}(1 - L_t)^{-\beta}$, we have:

$$\begin{aligned} g_\delta &= \lim_{t\to\infty} \dot{\ln}(\overline{\delta} A_t^{\epsilon-\beta} L_t^{\epsilon}(1 - L_t)^{-\beta}) \\ &= \lim_{t\to\infty} \Big[(\epsilon - \beta)\dot{\ln}(A_t) + \epsilon\dot{\ln}(L_t) - \beta\dot{\ln}(1 - L_t)\Big] \\ &= (\epsilon - \beta)g - \beta g_S \end{aligned} \tag{85}$$

where $\dot{\ln}(L_t) = g_L = 0$, as noted above.

Recall from (61) that

$$\frac{S_t}{L_t} = \frac{\tilde{v}_t \delta_t \beta}{1 - \tilde{v}_t \delta_t \epsilon}.$$

Since the denominators of this expression are asymptotically constant, and since β is of course a constant, S and $\tilde{v}\delta$ must in the limit shrink at the same rate. That is,

$$\begin{aligned} &\lim_{t\to\infty} g_{St} = g_{\tilde{v}_t \delta_t} \\ \implies\ & g_S = g_{\tilde{v}} + g_\delta \end{aligned} \tag{86}$$

on the asymptotic growth path.

The constraints that $\epsilon \leq \beta$ and $\gamma \leq 1 + \beta - \epsilon$ allow for $\gamma > 1$ or $\gamma \leq 1$.

When $\gamma > 1$, from we know from (95) that $g_{\tilde{v}t} \to (\gamma - 1)g_C$ along a balanced growth path in which $C_t \to \infty$, as long as δ_t converges to some constant.

From (86), (84), and (85) we therefore have

$$\begin{aligned} & g_S = (\gamma - 1)g + (\epsilon - \beta)g - \beta g_S \\ \implies\ & g_S = -g\frac{1 + \beta - \epsilon - \gamma}{1 + \beta}, \end{aligned} \tag{87}$$

which as we can see is negative if $\gamma < 1 + \beta - \epsilon$ and zero if $\gamma = 1 + \beta - \epsilon$.

From (84) and (87), we have

$$\begin{aligned} g_H &= g - g\frac{1 + \beta - \epsilon - \gamma}{1 + \beta} \\ &= g\frac{\gamma + \epsilon}{1 + \beta}. \end{aligned} \tag{88}$$

Note that, if $\gamma = 1 + \beta - \epsilon$ (the case where $g_S = 0$ given $\gamma > 1$), we have $g_H = g$, not $g_H = 0$. It follows that in this case $S_t \to S^* > 0$, and likewise $L_t \to L^* < 1$.

Finally, from (85) and (87), we have

$$\begin{aligned} g_\delta &= (\epsilon - \beta)g + \beta g\frac{1 + \beta - \epsilon - \gamma}{1 + \beta} \\ &= -g\frac{\beta\gamma - \epsilon}{1 + \beta}, \end{aligned} \tag{89}$$

which is negative given $\epsilon \leq \beta$, because $\gamma > 1$ in this case. So $\delta_t \to 0$.

When $\gamma \leq 1$,

$$\begin{aligned} g_{\tilde{v}} &= g_{\frac{u(C_t)}{u'(C_t)C_t}} - g_{\rho + \delta_t - g_{vt}} \\ &= g_{\overline{u}C_t^{\gamma-1} + \frac{1}{1-\gamma}} \\ &= 0, \end{aligned} \tag{90}$$

as long as δ_t converges to some constant.

From (86), (84), and (85) we therefore have

$$\begin{aligned} & g_S = (\epsilon - \beta)g - \beta g_S \\ \implies & g_S = -g\frac{\beta - \epsilon}{1 + \beta}, \end{aligned} \tag{91}$$

which as we can see is negative if $\epsilon < \beta$ and zero if $\epsilon = \beta$.

From (84) and (91), we have

$$\begin{aligned} g_H &= g - g\frac{\epsilon - \beta}{1 + \beta} \\ &= g\frac{1 + \epsilon}{1 + \beta}. \end{aligned} \tag{92}$$

Note that, if $\epsilon = \beta$ (the case where $g_S = 0$ given $\gamma \leq 1$), we again have $g_H = g$, not $g_H = 0$. It follows that in this case $S_t \to S^* > 0$, and likewise $L_t \to L^* < 1$.

Finally, from (85), and (91), we have

$$g_\delta = (\epsilon - \beta)g + \beta g \frac{\beta - \epsilon}{1 + \beta}$$
$$= -g \frac{\beta(2 + \beta) - \epsilon}{1 + \beta}, \tag{93}$$

which is negative given $\epsilon \leq \beta$.

A.5 Proof of Proposition 4

Note that the formulas for $g_C^*, g_L^*, g_H^*, g_S^*$, and g_δ^*, and the corresponding inequalities, are identical in this case as under Proposition 2. The proof is identical as well (including the section on the minimum valid value of ρ). This is because the key results that $g_L^* < 0$ and $g_C^* > 0$ follow from $\gamma > 1 + \beta - \epsilon$, which holds whenever $\epsilon > \beta$ and $\gamma > 1$.

A.6 Proof of Proposition 5

Let us conjecture that when $\epsilon > \beta$ and $1-\epsilon < \gamma \leq 1$, there is an asymptotic balanced growth path such that $L_t \to 0$, $g^*_C > 0$, and δ converges to a constant.

Observe that when $\gamma < 1$, given equations (68) and (67), along a balanced growth path in which $C_t \to \infty$:

$$\begin{aligned} g_{\tilde{v}} &= g_{\frac{u(C_t)}{u'(C_t)C_t}} - g_{\rho+\delta_t-g_{vt}} \\ &= g_{\overline{u}C_t^{\gamma-1}+\frac{1}{1-\gamma}} \\ &= 0 \end{aligned} \tag{94}$$

as long as δ_t converges to some constant. In the knife-edge case where $\gamma = 1$, $\frac{u(C_t)}{u'(C_t)C_t} = \bar{u}+ln(C_t)$. From this it follows that $g_{\frac{u(C_t)}{u'(C_t)C_t}} = \frac{g_c}{\bar{u}+ln(C_t)}$. Asymptotically, g_C is constant, while $C_t \to \infty$. Therefore, along the balanced growth path, $g_{\frac{u(C_t)}{u'(C_t)C_t}} = 0$ even when $\gamma = 1$.

Recall from (61) that

$$\frac{S_t}{L_t} = \frac{\tilde{v}_t\delta_t\beta}{1-\tilde{v}_t\delta_t\epsilon}. \tag{95}$$

Since $L_t \to 0$ and $S_t \to 1$ on the conjectured balanced growth path, the denominator of the right hand side must also tend to zero on the conjectured balanced growth path. Therefore, $\delta_t\tilde{v}_t \to 1/\epsilon$. This implies that $g^*_\delta = -g_{\tilde{v}} \Rightarrow g^*_\delta = 0$.

To find the asymptotic growth rate of L_t, consider

$$g_{\delta t} = (\epsilon - \beta)g + \epsilon g_{Lt} - \beta g_{St} \tag{96}$$

which follows directly from the construction of δ_t. Along the conjectured balanced growth path, $g^*_\delta = g^*_S = 0$. Therefore $g^*_L = g(\beta - \epsilon)/\epsilon$, which is less than zero, as conjectured.

Finally, to find the asymptotic value of δ_t, consider that as $t \to \infty$, $\delta_t \to 1/(\epsilon\tilde{v}_t)$. So

$$\begin{aligned} \delta^* &= \lim_{t\to\infty} \frac{1}{\epsilon}\frac{\rho+\delta_t-g_v}{\tilde{u}_t} = \frac{1}{\epsilon}\frac{\rho+\delta^*-(1-\gamma)g^*_C}{\frac{1}{1-\gamma}} \\ \implies \delta^* &= \frac{(1-\gamma)\rho-(1-\gamma)^2g^*_C}{\epsilon-1+\gamma}. \end{aligned} \tag{97}$$

As $1-\epsilon < \gamma$, the denominator is positive. The numerator is positive iff $\rho > (1-\gamma)g_c^*$. This ensures that our integral over utility is bounded: $u(C_t)$ grows at rate $(1-\gamma)g_C^*$ asymptotically, so if ρ were smaller than that, the integral would be unbounded, and the expected-utility-maximization problem would be undefined. Thus, δ does indeed converge to a constant as conjectured.

Note that when $\gamma = 1$, $\delta^* = 0$. Nevertheless, $g_\delta^* = 0$ as before. Since δ still does not fall exponentially, $M_\infty = 0$.

A.7 Proof of Proposition 6

Let us conjecture that when $\epsilon > \beta$ and $\gamma < 1-\epsilon$, there is an asymptotic balanced growth path such that L_t and S_t converge to positive constants, $g_C^* > 0$, and δ_t grows exponentially.

$$\begin{aligned} L_t \to L^* > 0 \text{ and } S_t \to S^* > 0 \\ \Rightarrow\; g_L^* = g_S^* = 0, \\ g_C^* = g_H^* = g, \\ g_\delta^* = (\epsilon - \beta) g. \end{aligned}$$

These expressions follow directly from the construction of C_t, H_t, and δ_t. All that remains is to check $\delta_t \tilde{v}_t < 1/\epsilon$ and to find L^* and S^*.

First, check $\delta_t \tilde{v}_t < 1/\epsilon$. Given $g_C^* > 0$ and $\gamma < 1-\epsilon$, we have $\frac{u(C_t)}{u'(C_t)C_t} \to 1/(1-\gamma)$. Recall that

$$\delta_t \tilde{v}_t = \frac{\delta_t}{\rho + \delta_t - g_{v_t}} \cdot \frac{u(C_t)}{u'(C_t)C_t}. \tag{98}$$

On an asymptotic growth path, g_{vt} is constant. ρ is also constant. Therefore, $\delta_t/(\rho + \delta_t - g_{vt}) \to 1$ as $t \to \infty$. Therefore,

$$\delta_t \tilde{v}_t \to \frac{1}{1-\gamma}. \tag{99}$$

Note that $\gamma < 1-\epsilon \Rightarrow 1/\epsilon > 1/(1-\gamma)$. So $\delta_t \tilde{v}_t < 1/\epsilon$ as required.

Recall from (61) that

$$\begin{aligned} \frac{S_t}{L_t} = \frac{1-L_t}{L_t} &= \frac{\beta \tilde{v}_t \delta_t}{1 - \epsilon \tilde{v}_t \delta_t} \\ \Rightarrow \frac{1-L^*}{L^*} &= \frac{\beta \frac{1}{1-\gamma}}{1 - \frac{\epsilon}{1-\gamma}} \\ \Rightarrow L^* = \frac{1-\gamma-\epsilon}{\beta + 1 - \gamma - \epsilon} &\Rightarrow S^* = \frac{\beta}{\beta + 1 - \gamma - \epsilon}. \end{aligned}$$

Given $\gamma < 1-\epsilon$, $1-\gamma-\epsilon > 0$ so L^* and S^* are both positive and less than one as required.

A.8 Laws of motion for L_t and δ_t

Note that for any variable a, $\widehat{(1-a)} = \frac{\dot{1-a}}{1-a} = -\frac{\dot{a}}{a}\frac{a}{1-a} = -\hat{a}\frac{a}{1-a}$. From this and the construction of A_t, C_t, and δ_t it follows that

$$\begin{aligned} g_{At} &= g, \\ g_{Ct} &= g + g_{Lt}, \\ g_{\delta t} &= (\epsilon - \beta)g + (\epsilon + \beta\frac{L_t}{1-L_t})g_{Lt}. \end{aligned}$$

From the construction of $u(C_t)$ we have that

$$g_{u'(C_t)C_t} = (1-\gamma)g_{Ct}.$$

Taking logs and derivatives of (61) we have that

$$\frac{-g_{Lt}}{1-L_t} = (g_{\delta t} + g_{vt} + (\gamma-1)g_{Ct})(1 + \frac{1-L_t}{L_t}\frac{\epsilon}{\beta}).$$

After some manipulation and substituting in the expression for g_{vt} from (63), this yields

$$g_{Lt} == \frac{(\epsilon-\beta)g + \rho - \frac{u(C_t)}{v_t} + \delta_t}{-1-\epsilon-\beta-(\gamma-1)\frac{1-L_t}{L_t}}.$$

To find an expression for $u(C_t)/v_t$, consider again (61), and see that:

$$\begin{aligned} \frac{u(C_t)}{v_t}(1-\epsilon\delta_t v_t C_t^{\gamma-1}) &= \frac{L_t}{1-L_t}\beta\delta_t C_t^{\gamma-1} \\ \Rightarrow \frac{u(C_t)}{v_t} &= \frac{L_t}{1-L_t}\beta\delta_t C_t^{\gamma-1}u(C_t) + \epsilon\delta_t C_t^{\gamma-1}u(C_t). \end{aligned}$$

As $C_t = A_t L_t$, and $u(C_t) = \bar{u} + \frac{C_t^{1-\gamma}}{1-\gamma}$ we have expressions for g_{Lt}, and $g_{\delta t}$ in terms of L_t, δ_t and A_t.