Labor, Capital, and Patience in the Optimal Growth of Social Movements

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1 Introduction

Social movements, or those organizing them, must allocate their resources across time in order to maximize their desired impact. Their two principal resources might naturally be classified as labor (movement participants) and capital (funds given in support of the movement). The animal welfare movement, for instance, spends time and money not only on immediately generating its desired "products"—such agricultural monitoring and rescue shelters—directed to the prevention and relief of animal suffering, but also on financial investment, and outreach to potential further movement participants, in order to grow the stocks of capital and labor available for producing what is necessary for the prevention and relief of animal suffering in the future. More abstractly, the Effective Altruism (EA) movement sets itself the broad goal of doing as much good as possible, across all domains, and its members must determine how to allocate their resources between efforts to do good in the present and investment and outreach efforts that allow for more future do-gooding.

Framed this way, a movement's intertemporal resource allocation problem is in some ways analogous to that of a social planner who must set policies affecting the social allocation of labor and capital over time, in a setting of exogenous technological development. Whereas a nation's population dynamics are usually considered exogenous, however—or perhaps directly controllable through immigration policy—social movements grow by explicit movement-building efforts, i.e. by investing labor and capital into the recruitment of like-minded associates. They can also effectively convert their labor to capital by asking their members not to work directly on projects that further the aims of the movement but to earn market wages and contribute a

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portion of these to the cause; likewise, they can sometimes convert their capital to labor by hiring people outside the movement to work on movement-relevant projects. Finally, whereas a planner's objective is generally understood to be the welfare of her society, somehow defined, a movement's objective is not in general the welfare of its members. The movement's goals are thus defined only with respect to its total production of impact (however construed), not on anything analogous to production per capita.

We here model the optimal intertemporal allocation of capital and labor for a social movement. At every period, the movement can put its resources to use in two ways: in "direct work", modeled as the production of a good over which the movement's utility function is defined, and in "recruitment", meaning the acquisition or retention of movement participants.

§2 explores the long-run dynamics of a model in which capital can be spent in direct work, spent in recruitment, or invested for future use, and in which labor can be used in direct work or in recruitment.

§3 explores a model like that of §2, but in which capital can also be used to hire labor from outside the movement, and labor can also be used to "earn to give" capital to the movement. In this model, in which the two factors are more fungible, we can calculate not only the movement's long-run dynamics but also its transition dynamics.

In §4, we discuss the implications of the two models, in combination, for social movements in which earning to give is feasible but hiring is not. We also discuss implications for the EA movement in particular.

2 Model without earning to give or hiring

Throughout the models of both this section and the following section, we will model the movement as an agent, with flow utility isoelastic in the production of its "direct work". The movement can spend its capital on direct work or recruitment, or it can invest it at a constant, exogenous interest rate r. Labor can be used in direct work or recruitment, and it depreciates at exogenous rate $d > 0$, representing the rate at which members die, retire, or exit. d thus equals the rate at which the movement's population shrinks without sustained efforts at recruitment, retention, and intergenerational values transmission (which we will collectively refer to as "recruitment").[1]

Labor productivity in direct work is defined to equal "B_D" (> 0) at time zero and assumed to grow at exogenous rate $\gamma \in [0, r)$. Production is assumed to exhibit

[1] Nonpositive values of d would represent cases in which the movement's membership on net is stable or grows, perhaps by attracting or parenting new members, even without expending resources explicitly on recruitment efforts or on the intergenerational transmission of values. We intend to extend these analyses to cases of $d \leq 0$.

constant returns to scale in capital and effective labor, and constant elasticity of substitution between the two factors. The recruitment of effective labor is assumed to be a concave transformation of a function that is likewise CRS and CES in capital and effective labor, and thus it is assumed to exhibit decreasing returns to scale overall. This may be interpreted as the assumption that the production of concrete recruitment materials—such as pamphlets, school lessons, or conversations—exhibits constant returns to scale, but that the marginal effectiveness of these materials diminishes as outreach at a time extends to less amenable potential recruits.

We would like to maintain the intuitive condition that recruiting effective labor does not grow easier (or harder) as labor productivity rises, i.e. that movement-builders must spend a constant number of hours per expected recruit. The difficulty of recruiting labor is thus also assumed to grow alongside labor productivity at rate γ.

Finally, labor is assumed to be paid a wage proportional to its productivity. If productivity begins at B_D, wages paid to labor hired from outside the movement begin at B_D, and wages paid to labor hired from within the movement begin at αB_D for some $\alpha \in [0, 1]$. Both wage rates grow alongside productivity at rate γ. Supposing labor productivity grows at the same rate within the movement as in the economy at large, this wage growth assumption can be given a natural interpretation. For labor from outside the movement—"outside" labor—it amounts to the assumption that wages for jobs sponsored by the movement must compete with wages elsewhere. For "inside" labor, it amounts to the assumption that, while movement members may be willing to do movement work at a wage discount, this discount is constant in proportional terms.

The movement begins at time 0 with capital and labor stocks K_0 and L_0 respectively. The movement organizer's problem is to allocate these resources over time so as to maximize the movement's utility over an infinite horizon, after discounting at time preference rate $\delta > 0$.

Throughout this section, we will further assume that capital cannot be used to hire or contract outside labor (which we will generically call "hiring"), and that labor cannot be used to earn and contribute capital (which we will call "earning to give") but can only do direct movement work. These restrictions will both be relaxed in the next section.

The first of these restrictions may be justified on the grounds that many of the activities in which a movement seeks to engage—especially recruitment efforts, but also some varieties of direct work—require idiosyncratic knowledge or motivation that cannot effectively be replicated by hired hands. Those tasks which can be entrusted to individuals outside the movement, furthermore, will often presumably be complementary to tasks, such as monitoring, which require the time of insiders. Hiring can thus implicitly be modeled as a capital expenditure, with the "labor" term in the production and recruitment functions reserved for inside labor.

The second restriction is not realistic. Earning to give, instead of engaging personally in direct work or recruitment activities, is common in many real-world social movements. At least under the simplifying assumptions explored in the following section, however, given the opportunity to hire or to encourage its members to earn to give, a movement optimally engages in at most one of these at a time. Which of these is optimal at a given time depends in an intuitive way on the movement's capital to labor ratio. As we will see, furthermore, because capital effortlessly grows with time at the interest rate whereas effective labor, on its own, presumably depreciates or grows more slowly, it seems most natural to suppose that on the optimal path a movement eventually grows wealthy enough to prefer hiring, rather than earning to give, when both options are available. It seems likely, therefore, that the restricted models of this section capture optimal behavior in the realistic case where earning to give is an option but hiring is not, and where hiring alone would be desirable.

Thus, in the model of this section, we have

$$U_t = u(D(k_{Dt}, \ell_{Dt} L_t e^{\gamma t})), \tag{1}$$

$$\dot{K}_t = rK_t - k_{Dt} - k_{Rt} - \alpha B_D e^{\gamma t} L_t, \tag{2}$$

$$\dot{L}_t = -dL_t + R(k_{Rt} e^{-\gamma t}/B_D, (1-\ell_{Dt})L_t), \tag{3}$$

where the flow utility function u is isoelastic with inverse elasticity of intertemporal substitution (IEIS) $\eta > 0$, i.e.

$$u(x) \triangleq \begin{cases} \frac{x^{1-\eta}-1}{1-\eta}, & \eta \neq 1, \\ \ln(x), & \eta = 1; \end{cases} \tag{4}$$

the direct work production function D is CRS and CES; and the recruitment function R is a concave transformation of a function that is CRS and CES. k_D and k_R represent flow expenditures on direct work and recruitment, and ℓ_D represents the fraction of labor doing direct work rather than recruitment.

In this section, let us also assume that capital and labor are gross complements in both direct work and recruitment. In particular, assume

$$D\big(k_{Dt}, \ell_{Dt} L_t e^{\gamma t}\big) = \Big[\big(A_D k_{Dt}\big)^{\rho_D} + \big(B_D \ell_{Dt} L_t e^{\gamma t}\big)^{\rho_D}\Big]^{\frac{1}{\rho_D}}, \tag{5}$$

$$R\big(k_{Rt} e^{-\gamma t}/B_D, (1-\ell_{Dt})L_t\big) = \Big[\big(A_R k_{Rt} e^{-\gamma t}/B_D\big)^{\rho_R} + \big(B_R (1-\ell_{Dt}) L_t\big)^{\rho_R}\Big]^{\frac{\lambda}{\rho_R}}, \tag{6}$$

where $\rho_D < 0$, $\rho_R < 0$, $\lambda \in (0,1)$, and A_D, B_D, A_R, and B_R (all > 0) represent capital and labor productivity in direct work and recruitment respectively.

Note that we have set B_D simultaneously equal to the initial wage and to the initial level of labor productivity in direct work. Because the units of direct work are arbitrary, this is without loss of generality.

Two of the four productivity terms could also be eliminated, without loss of generality, by denominating labor and capital such that the respective terms equal one. However, it is convenient to retain all four so that labor and capital can be denominated in familiar units for calibration purposes, and to clarify the relationships between factor productivities and the nature of the results below.

Definition 1. *A movement is* ***patient*** *if*

$$\delta < r - \gamma\eta. \tag{7}$$

Proposition 1. *In the model above, a patient movement approaches a growth path in which the labor stock and labor allocation are constant at*

$$L = d^{\frac{1}{\lambda-1}} \left(B_R \lambda\right)^{\frac{\lambda}{1-\lambda}}, \tag{8}$$

$$\ell_D = 1 - \lambda; \tag{9}$$

spending grows at rates

$$g_{k_D} = \frac{r - \delta + \gamma(1 - \eta - \rho_D)}{1 - \rho_D}, \tag{10}$$

$$g_{k_R} = \frac{r - \delta + \gamma(1 - \eta - \rho_R)}{1 - \rho_R}; \tag{11}$$

and the capital stock grows at rate

$$g_K = \max(g_{k_D}, g_{k_R}). \tag{12}$$

Proof. See Appendix A.1. □

Let us make four observations about this result.

First, the labor stock does grow in the long run, but approaches a finite value. This is because the necessity of labor for recruitment, and the diminishing returns to recruitment, imply that the movement's membership cannot grow indefinitely; even if all labor were spent on recruitment, and even if infinite capital were available, membership would grow only to a steady state in which the attrition and recruitment rates were equal.

Relatedly, observe that as the recruitment function approaches constant returns to scale ($\lambda = 1$), the long-run proportional allocation of labor to recruitment approaches 1.

Of course, opinions may differ about whether any given movement is naturally constrained in this way. The model we subsequently explore will therefore relax the condition that sustained growth in the recruitment labor force is necessary for sustained growth in recruitment, thereby allowing for unbounded movement membership despite diminishing returns to recruitment activity.

Second, by the Ramsey Formula, we should expect the economic growth rate to equal $(r - \tilde{\delta})/\tilde{\eta}$, where $\tilde{\delta}$ and $\tilde{\eta}$ represent a typical household's time preference rate and IEIS. Furthermore, by Uzawa's Theorem, we should expect the economic growth rate to equal the overall labor productivity growth rate γ (which, recall, we have been assuming is equal for the movement as for the economy on the whole). If a social movement's time preference rate and EIS are equal to those faced by the society at large, therefore, (7) fails; the relation holds with equality. If $\delta < \tilde{\delta}$ however—i.e. if the movement is atypically patient, but otherwise exhibits typical parameter values—then (7) does hold.

Third, it is only in the knife-edge case of $\rho_D = \rho_R$ that $g_{k_D} = g_{k_R}$. If $\rho_D > \rho_R$, then $g_{k_D} > g_{k_R}$, and vice-versa. Thus, even though the labor allocation approaches a constant strictly between 0 and 1, spending in the long run is typically focused almost entirely either on production or on recruitment: whichever domain allows capital better to substitute for labor.

Finally, if a movement's only resource were capital, and if flow utility were isoelastic in spending alone rather than in the output of a two-factor production function, then the movement's intertemporal resource allocation problem would be equivalent to a household's consumption smoothing problem, and its spending and capital growth rates would equal $(r - \delta)/\eta$. We might therefore wish to determine whether capital growth is greater or less in this labor-constrained case than in the capital-only case.

It follows from (10)–(11) that $g_X \geq (r-\delta)/\eta$ precisely when $\rho_X \geq 1-\eta$, and vice-versa, for $X = D, R$. Thus, when $\max(\rho_D, \rho_R) = 1 - \eta$, the movement's funds grow just as quickly as in the capital-only case. Interestingly, with common estimates of household IEIS around 1.5, and estimates of capital-labor substitution parameters in production typically in the vicinity of –0.5, it is reasonable in the absence of further evidence to expect spending growth not to be particularly faster or slower—i.e. for proportional spending rates not to be particularly smaller or larger—for a labor-constrained movement of the kind outlined here than for one concerned only with the allocation of capital.

3 Model with earning to give and hiring

Now let us consider a model in which movement participants can take work outside the movement and contribute fraction $1-\alpha$ of their income to the movement, instead of working for the movement at wage discount α, and in which labor from outside the movement can be hired at the full going wage.

We thus have

$$\dot{K}_t = rK_t - k_{Dt} - k_{Rt} + B_D L_t e^{\gamma t}(1 - \ell_{Dt} - \ell_{Rt} - \alpha) \text{ and} \tag{13}$$

$$\dot{L}_t = -dL_t + R(k_{Rt}e^{-\gamma t}/B_D, \ell_{Rt}L_t) \tag{14}$$

in place of (2) and (3) from the previous section.

Let us also remove our restrictions on ρ_D and ρ_L. We will now assume only that $\rho_D < 1$ and $\rho_R < 1$, not that these terms are negative. That is, we will relax the gross complementarity assumptions. Otherwise, we will retain the model as described in terms (1)–(6) of the previous section.[2]

Allowing earning to give and hiring allows us to derive an exact solution to the dynamic optimization problem, not just an asymptotic result.

Proposition 2. *In the model above, a movement optimally allocates its resources as follows:*

$$
\begin{aligned}
k_{Dt} &= k_{D0} e^{\frac{r-\delta}{\eta} t}, && (15)\\
k_{Rt} &= k_{R0} e^{\gamma t}, && (16)\\
\ell_{Dt} &= k_{D0} A_D^{\frac{\rho_D}{\rho_D - 1}} \frac{1}{B_D L_t} e^{\left(\frac{r-\delta}{\eta} - \gamma\right) t}, && (17)\\
\ell_{Rt} &= k_{R0} \frac{C_1 - 1}{B_D L_t}, && (18)\\
K_t &= k_{D0}\left(1 + A_D^{\frac{\rho_D}{\rho_D - 1}}\right) \frac{\eta}{r\eta - r + \delta} e^{\frac{r-\delta}{\eta} t} && (19)\\
&\quad + \left(k_{R0} C_1 - B_D(1-\alpha) C_2\right) \frac{1}{r - \gamma} e^{\gamma t}\\
&\quad + B_D(1-\alpha)(L_0 - C_2) \frac{\gamma - d}{\gamma - d - r} e^{(\gamma - d)t}, \textit{ and}\\
L_t &= (L_0 - C_2) e^{-dt} + C_2, && (20)
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &\triangleq 1 + \left(\frac{A_R}{B_R}\right)^{\frac{\rho_R}{\rho_R - 1}}, && (21)\\
C_2 &\triangleq \frac{1}{d}\left(\frac{A_R}{B_D} \frac{r - \gamma + d}{\lambda(1-\alpha)}\right)^{\frac{\lambda}{1-\lambda}} C_1^{\frac{\lambda(1-\rho_R)}{\rho_R(1-\lambda)}}, && (22)\\
k_{R0} &= \left(\frac{\lambda(1-\alpha)}{r - \gamma + d}\right)^{\frac{1}{1-\lambda}} \left(\frac{A_R}{B_D}\right)^{\frac{\lambda}{1-\lambda}} C_1^{\frac{\lambda - \rho_R}{\rho_R(1-\lambda)}}, \textit{ and} && (23)\\
k_{D0} &= \left[K_0 - \frac{k_{R0}}{r - \gamma} C_1 + B_D(1-\alpha)\left(\frac{L_0 - C_2}{d + r - \gamma} + \frac{C_2}{r - \gamma}\right)\right]\\
&\quad \cdot \frac{r\eta - r + \delta}{\eta} A_D^{\frac{1-\eta}{\eta}} \left(1 + A_D^{\frac{\rho_D}{1-\rho_D}}\right)^{-1}. && (24)
\end{aligned}
$$

[2]If $\rho_D = 0$ or $\rho_R = 0$, function D or R, as described by (5) or (6) respectively, must be replaced by its Cobb-Douglas equivalent. We intend to solve the Cobb-Douglas cases of this model eventually as well, but for now the results presented here assume $\rho_D \neq 0$ and $\rho_R \neq 0$. None of the qualitative results discussed below exhibit phase changes around $\rho_D = 0$ or $\rho_R = 0$.

Proof. See Appendix A.2. □

As we can see, as in the model of the previous section, the movement approaches a constant size in the long run. Here, that size is given by C_2. Likewise, though less obviously, the fraction of the movement engaged in recruitment ℓ_R here always approaches a constant: here equal not to λ but to

$$\lambda \frac{d}{B_D} \frac{1-\alpha}{r-\gamma+d} \left(1 + \left(\frac{A_R}{B_R}\right)^{\frac{\rho_R}{1-\rho_R}}\right)^{-1}. \tag{25}$$

The absolute number of people engaged in recruitment is thus constant in the long run.

Also as in the model of the previous section, however, long-run movement dynamics here depend crucially on whether the movement is patient, i.e. on whether it satisfies (7).

If it does, then k_D grows more quickly than k_R, so the share of spending allocated to direct work approaches 1 and that allocated to recruitment approaches 0. Also, in this case ℓ_D exhibits a positive growth rate. That is, the labor force assigned to direct work divided by the total number of individuals in the movement rises to infinity. Of course, this implies that, in the long run, everyone or almost everyone doing direct work is hired, not drawn from movement ranks.

If the movement is atypically impatient, i.e. if the inequality of (7) is reversed, then k_R grows more quickly than k_D and ℓ_D exhibits a negative growth rate. Since the number of movement members approaches a finite constant C_2, this implies that the absolute number of people doing direct work falls to zero (though not as quickly as their productivity rises).

This may at first seem paradoxical. An impatient movement might be expected to spend its resources in direct work to the neglect of "investment" in recruitment, but here we find the reverse. An intuition for the result is as follows. Because of the possibility of earning to give and hiring, the value of movement labor is always equal to α times the net present value of the wages this labor will earn. The optimal allocations of capital and labor to recruitment, k_{Rt} and $\ell_{Rt}L_t$, are simply those that leave the marginal value of "investing in recruitment" equal to r. They are independent of δ; any other allocations would leave money on the table. With ℓ_R constant and ℓ_D falling to zero, fraction of movement participants earning to give tends to $1-\ell_R$. The stream of revenues produced by these earnings are then shifted to present direct-work spending by borrowing. The starting spending rate k_{D0} is increasing in δ, but the growth rate of k_D is $(r-\delta)/\eta$, which is here less than γ, the growth rate of revenues from earners.

Thus, the optimal path described here is incompatible with a "no-borrowing" constraint that

$$K_t \geq 0 \quad \forall t \geq 0, \tag{26}$$

for impatient movements. Without imposing such a constraint, the model of this section is thoroughly unrealistic. It suggests that an impatient movement will be able to recruit members in perpetuity to pay off the debt of an early splurge, while the number of individuals actually engaged in movement activity falls to zero.

By contrast, for non-impatient movements (i.e. if $\delta \leq r - \gamma\eta$), (26) may or may not be temporarily violated early in time. However, it will always be satisfied in the long run. Borrowing under these circumstances is arguably realistic. It is not uncommon for churches to fund expansions by borrowing, for instance, on the understanding that larger congregations will collect the revenues necessary to repay the loan. Thus, as long as the movement institution has healthy long-term prospects, it may sometimes be natural to observe temporary periods of at least moderate borrowing.

Finally, in the knife-edge case that $\gamma = (r - \delta)/\eta$, k_R and k_D grow at the same rate. The constant ratio between the two is simply the ratio between (23) and (24). ℓ_D and ℓ_R, and thus also the absolute numbers of people engaged direct work and recruitment, are both constant in the long run.

4 Discussion

A patient movement for which both earning to give and hiring are feasible may wish to engage in earning to give (and not hiring), if it begins with sufficiently abundant labor and scarce capital. In particular, for high L_0 and low K_0, we see that $\ell_{D0} + \ell_{R0} < 1$. In the long run, however, the movement will always wish to engage in hiring (and not earning to give). If hiring is not feasible, then once the no-hiring constraint

$$\ell_{D0} + \ell_{R0} \leq 1 \tag{27}$$

binds—i.e. once (27) holds with equality—the movement is essentially characterized by the model of §2, as long as $\rho_D < 1$ and $\rho_R < 1$.

Ideally, we would simply solve the patient movement's optimization problem under constraint (27) (and potentially also (26)). However, this does not appear to be possible using standard dynamic optimization techniques. For now, therefore, we will just observe by the rough reasoning of §3 that a patient movement will, in the long run, not encourage its members to earn to give, and that it can therefore ultimately be described approximately by the model of §2.

Note that, under constraint (27), it is not simply optimal for the movement to follow the path given by Proposition 2 until (27) binds. This is because, for a movement for which both earning to give and hiring are feasible, earning to give early in time is partly motivated by the prospect of investing for the sake of future hiring. If hiring will never be feasible, therefore, this affects optimal movement strategy even before hiring is desirable.

The qualitative conclusion for the EA movement is straightforward.

In the movement's early days, it consisted primarily of students. It possessed relatively abundant human capital (high L_0) but relatively little other capital (low K_0). Earning to give was a valuable activity, and many members took high-paying jobs to fund the creation and maintenance of direct-work and recruitment institutions staffed primarily by other movement members. Over the years since, however, the stock of capital has grown more rapidly than the stock of labor, and the apparent value of earning to give—at least for spending on direct work or recruitment in the present—has fallen relative to the apparent value of labor contributions to the two sectors.

One possibility raised by these circumstances is that the movement's capital-to-labor ratio simply fluctuates over time. If so, the value of capital contributions relative to that of labor contributions fluctuates as well. Also, during a time with atypically plentiful capital per unit of labor, earning to give may be more valuable than it first appears, since funds can be invested for use when they are more needed.

In light of the analysis presented here, however, another possibility seems more likely. Because of the EA movement's rejection of pure time preference on ethical grounds, it seems likely that the movement satisfies condition (7). This suggests the possibility that, as long as the EA movement follows the optimal path, it has now entered a regime in which earning to give will always be of relatively little value. Perhaps from now on, the movement should, to a first approximation, be funded entirely by the interest on an endowment that receives no further contributions.

To be sure, the stylized models discussed here fail to capture a number of important features of EA movement dynamics. In particular, the extraordinary growth in EA capital in recent years has been generated largely not by interest on investments or by small donors contributing fractions of their incomes but by the recruitment of a handful of already-wealthy philanthropists. As a result, the movement reached the "long-run environment"—a high capital-to-labor ratio, with earning to give undesirable—more quickly than it would have otherwise. Having reached the "long run", however, the optimal path forward is likely the same. As long as capital and labor are gross complements in both direct work and recruitment (which seems likely to be the empirically relevant case), this optimal path is the one described asymptotically in Proposition 1.

Appendices

A Proofs

A.1 Proof of Proposition 1

Let g_{xt} denote the proportional growth rate of variable x at time t, and let $g_x \triangleq \lim_{t\to\infty} g_{xt}$, if this is defined.

L cannot grow without bound; even if all labor were used in recruitment, and even with infinite capital available, the steady-state labor pool would be finite. Thus if $g_{k_R} > \gamma$, we here, for large t, have approximately

$$\dot{L}_t = -dL_t + \big(B_R(1-\ell_{Dt})L_t\big)^{\lambda}. \tag{28}$$

Given constant ℓ_D, there is a steady-state labor stock L satisfying

$$\begin{aligned} 0 &= -dL + \big(B_L(1-\ell_D)L\big)^{\lambda} \\ \implies L &= d^{\frac{1}{\lambda-1}}\big(B_L(1-\ell_D)\big)^{\frac{\lambda}{1-\lambda}}. \end{aligned} \tag{29}$$

(There is also a steady-state labor stock of $L = 0$, but it is unstable and irrelevant for our purposes.)

The value of ℓ_D that maximizes steady-state labor in direct work, $\ell_D L$, then satisfies the first-order condition

$$\begin{aligned} 0 &= d^{\frac{1}{\lambda-1}} B_R^{-\frac{\lambda}{\lambda-1}}\left[(1-\ell_D)^{-\frac{\lambda}{\lambda-1}} + \frac{\lambda}{\lambda-1}\ell_D(1-\ell_D)^{-\frac{\lambda}{\lambda-1}-1}\right] \\ \implies 0 &= 1 + \frac{\lambda}{\lambda-1}\frac{\ell_D}{1-\ell_D} \\ \implies \ell_D &= 1-\lambda. \end{aligned} \tag{30}$$

Assuming also that $g_{k_D} > \gamma$, we thus have asymptotically

$$U_t = u\big(B_D(1-\lambda)Le^{\gamma t}\big) \tag{31}$$

$$\implies \frac{\partial U_t}{\partial[(1-\lambda)L]} = B_D^{1-\eta}((1-\lambda)L)^{-\eta}e^{\gamma(1-\eta)t}, \tag{32}$$

so the shadow value of labor at t is

$$\begin{aligned} & B_D^{1-\eta}((1-\lambda)L)^{-\eta}e^{\gamma(1-\eta)t}\int_0^\infty e^{[-d-\delta+\gamma(1-\eta)]s}ds \\ = & \frac{1}{\delta+d-\gamma(1-\eta)}B_D^{1-\eta}((1-\lambda)L)^{-\eta}e^{\gamma(1-\eta)t}. \end{aligned} \tag{33}$$

Now let us find the capital that must be spent to recruit marginal labor and compensate it for its direct work. Recruitment per unit of capital is given by

$$\begin{aligned}\frac{\partial \dot{L}_t}{\partial k_{Rt}} &= \lambda\Big[\big(A_R k_{Rt} e^{-\gamma t}/B_D\big)^{\rho_R} + \big(B_R \lambda L\big)^{\rho_R}\Big]^{\frac{\lambda}{\rho_R}-1}\big(A_R/B_D\big)^{\rho_R} k_{Rt}^{\rho_R-1} e^{-\gamma\rho_R t} \\ &\to \lambda\big(B_R\lambda L\big)^{\lambda-\rho_R}\big(A_R/B_D\big)^{\rho_R} k_{Rt}^{\rho_R-1} e^{-\gamma\rho_R t}. \end{aligned} \tag{34}$$

Inverting this term gives us capital per recruit. We must add the cost, in present value terms, of compensating this recruit for her employment in movement activity:

$$\alpha B_D e^{\gamma t}\int_0^\infty e^{(-r-d+\gamma)s} ds = \alpha B_D \frac{e^{\gamma t}}{d+r-\gamma}; \tag{35}$$

1/(34) + (35) equals

$$\frac{1}{\lambda}\big(B_R\lambda L\big)^{\rho_R-\lambda}\big(A_R/B_D\big)^{-\rho_R} k_{Rt}^{1-\rho_R} e^{\gamma\rho_R t} + \alpha B_D \frac{e^{\gamma t}}{d+r-\gamma}. \tag{36}$$

The utility produced by marginal labor, that has been recruited by capital, must equal the utility that could have been produced by that capital in direct work:

$$\begin{aligned}\frac{\partial U_t}{\partial k_{Dt}} &= \Big[\big(A_D k_{Dt}\big)^{\rho_D} + \big(B_D(1-\lambda)Le^{\gamma t}\big)^{\rho_D}\Big]^{\frac{1-\eta}{\rho_D}-1} A_D^{\rho_D} k_{Dt}^{\rho_D-1} \\ &\to \big(B_D(1-\lambda)Le^{\gamma t}\big)^{1-\eta-\rho_D} A_D^{\rho_D} k_{Dt}^{\rho_D-1}. \end{aligned} \tag{37}$$

Thus, setting (33) equal to (36) × (37), we have

$$\begin{aligned} &\frac{1}{\delta+d-\gamma(1-\eta)} B_D^{1-\eta}((1-\lambda)L)^{-\eta} e^{\gamma(1-\eta)t} \\ &= \Big(\frac{1}{\lambda}\big(B_R\lambda L\big)^{\rho_R-\lambda}\big(A_R/B_D\big)^{-\rho_R} k_{Rt}^{1-\rho_R} e^{\gamma\rho_R t} + \frac{\alpha B_D}{d+r-\gamma} e^{\gamma t}\Big) \\ &\cdot\big(B_D(1-\lambda)Le^{\gamma t}\big)^{1-\eta-\rho_D} A_D^{\rho_D} k_{Dt}^{\rho_D-1} \end{aligned} \tag{38}$$

$$\begin{aligned} \implies \quad &\frac{\big(A_D/B_D\big)^{-\rho_D}}{\delta+d-\gamma(1-\eta)} k_{Dt}^{1-\rho_D} \\ &= \frac{1}{\lambda}\big(B_R\lambda L\big)^{\rho_R-\lambda}\big(A_R/B_D\big)^{-\rho_R} k_{Rt}^{1-\rho_R} e^{\gamma(\rho_R-\rho_D)t} + \frac{\alpha B_D}{d+r-\gamma} e^{\gamma(1-\rho_D)t} \end{aligned} \tag{39}$$

$$\begin{aligned} \implies \quad (1-\rho_D) g_{k_D} &= \max\Big((1-\rho_R) g_{k_R} + \gamma(\rho_R-\rho_D), \gamma(1-\rho_D)\Big) \\ &= (1-\rho_R) g_{k_R} + \gamma(\rho_R-\rho_D), \end{aligned} \tag{40}$$

with the last step following from the assumption that $g_{k_R} > \gamma$.

Then by (37), to satisfy the movement's intertemporal Euler equation for capital we must asymptotically have

$$\begin{aligned} r &= \delta + (1-\rho_D)g_{k_D} - \gamma(1-\eta-\rho_D) \\ \implies g_{k_D} &= \frac{r-\delta+\gamma(1-\eta-\rho_D)}{1-\rho_D} \\ \implies g_{k_R} &= \frac{r-\delta+\gamma(1-\eta-\rho_R)}{1-\rho_R}. \end{aligned} \tag{41}$$

Finally, we maintain the conditions that $g_{k_D} > \gamma$ and $g_{k_R} > \gamma$ iff

$$\delta < r - \gamma\eta. \tag{42}$$

A.2 Proof of Proposition 2

This is our problem setup, and current value Hamiltonian, in terms of raw labor:

$$U_t = u\Big(\Big((A_D k_{Dt})^{\rho_D} + (L_t \ell_{Dt} B_D e^{\gamma t})^{\rho_D}\Big)^{\frac{1}{\rho_D}}\Big) \tag{43}$$

$$\dot{K}_t = rK_t - k_{Dt} - k_{Rt} + L_t B_D e^{\gamma t}(1 - \ell_{Dt} - \ell_{Rt} - \alpha) \tag{44}$$

$$\dot{L}_t = -dL_t + \Big((A_R k_{Rt} e^{-\gamma t}/B_D)^{\rho_R} + (B_R L_t \ell_{Rt})^{\rho_R}\Big)^{\frac{\lambda}{\rho_R}} \tag{45}$$

$$H_t = U_t + \mu_{Kt}\dot{K}_t + \mu_{Lt}\dot{L}_t \tag{46}$$

u is the isoelastic function with IEIS η, for some $\eta > 0$:

$$u(x) \triangleq \begin{cases} \frac{x^{1-\eta}-1}{1-\eta}, & \eta \neq 1, \\ \ln(x), & \eta = 1. \end{cases} \tag{47}$$

This is therefore our problem setup, and current value Hamiltonian, in terms of effective (productivity-adjusted) labor:

$$E_t \triangleq L_t e^{\gamma t} \tag{48}$$

$$U_t = u\Big(\Big((A_D k_{Dt})^{\rho_D} + (B_D E_t \ell_{Dt})^{\rho_D}\Big)^{\frac{1}{\rho_D}}\Big) \tag{49}$$

$$\dot{K}_t = rK_t - k_{Dt} - k_{Rt} + B_D E_t(1 - \ell_{Dt} - \ell_{Rt} - \alpha) \tag{50}$$

$$\dot{E}_t = (\gamma - d)E_t + B_D^{-\lambda} e^{\gamma(1-\lambda)t}\Big((A_R k_{Rt})^{\rho_R} + (B_R B_D E_t \ell_{Rt})^{\rho_R}\Big)^{\frac{\lambda}{\rho_R}} \tag{51}$$

$$H_t = U_t + \mu_{Kt}\dot{K}_t + \mu_{Lt}\dot{E}_t \tag{52}$$

$$H_t = \frac{1}{1-\eta}\Big((A_D k_{Dt})^{\rho_D} + (B_D E_t \ell_{Dt})^{\rho_D}\Big)^{\frac{1-\eta}{\rho_D}} \tag{53}$$

$$
\begin{aligned}
&+ \mu_{Kt}[rK_t - k_{Dt} - k_{Rt} + B_D E_t(1 - \ell_{Dt} - \ell_{Rt} - \alpha)] \\
&+ \mu_{Lt}\Big[(\gamma - d)E_t + B_D^{-\lambda} e^{\gamma(1-\lambda)t}\Big((A_R k_{Rt})^{\rho_R} + (B_R B_D E_t \ell_{Rt})^{\rho_R}\Big)^{\frac{\lambda}{\rho_R}}\Big]
\end{aligned}
$$

Our Hamiltonian first-order conditions give us:

$$
\begin{aligned}
&\frac{\partial H_t}{\partial k_{Dt}} = 0 \\
\implies\ &\mu_{Kt} = A_D^{\rho_D} k_{Dt}^{\rho_D - 1}\Big((A_D k_{Dt})^{\rho_D} + (B_D E_t \ell_{Dt})^{\rho_D}\Big)^{\frac{1-\eta}{\rho_D} - 1}; \qquad (54)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial H_t}{\partial k_{Rt}} = 0 \\
\implies\ &\mu_{Kt} = \mu_{Lt} B_D^{-\lambda} e^{\gamma(1-\lambda)t} \lambda A_R^{\rho_R} k_{Rt}^{\rho_R - 1}\Big((A_R k_{Rt})^{\rho_R} + (B_R B_D E_t \ell_{Rt})^{\rho_R}\Big)^{\frac{\lambda}{\rho_R} - 1}; \qquad (55)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial H_t}{\partial \ell_{Dt}} = 0 \\
\implies\ &\mu_{Kt} = (B_D E_t \ell_{Dt})^{\rho_D - 1}\Big((A_D k_{Dt})^{\rho_D} + (B_D E_t \ell_{Dt})^{\rho_D}\Big)^{\frac{1-\eta}{\rho_D} - 1}; \qquad (56)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial H_t}{\partial \ell_{Rt}} = 0 \qquad (57) \\
\implies\ &\mu_{Kt} = \mu_{Lt} B_D^{-\lambda} e^{\gamma(1-\lambda)t} \lambda B_R^{\rho_R} (B_D E_t \ell_{Rt})^{\rho_R - 1}\Big((A_R k_{Rt})^{\rho_R} + (B_R B_D E_t \ell_{Rt})^{\rho_R}\Big)^{\frac{\lambda}{\rho_R} - 1}; \qquad (58)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\partial H_t}{\partial K_t} = \dot{\mu_{Kt}} - \delta\mu_{Kt}, \quad \frac{\partial H_t}{\partial K_t} = r\mu_{Kt} \\
\implies\ &-r\mu_{Kt} = \dot{\mu_{Kt}} - \delta\mu_{Kt} \\
\implies\ &\mu_{Kt} = c_1 e^{(\delta - r)t}; \qquad (59)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\partial H_t}{\partial E_t} = \dot{\mu_{Lt}} - \delta\mu_{Lt}, \qquad (60) \\
&\frac{\partial H_t}{\partial E_t} = (B_D \ell_{Dt})^{\rho_D} E_t^{\rho_D - 1}\Big((A_D k_{Dt})^{\rho_D} + (B_D E_t \ell_{Dt})^{\rho_D}\Big)^{\frac{1-\eta}{\rho_D} - 1} \qquad (61) \\
&\quad + \mu_{Kt} B_D(1 - \ell_{Dt} - \ell_{Rt} - \alpha) \\
&\quad + \mu_{Lt}(\gamma - d)
\end{aligned}
$$

$$+\,\mu_{Lt}B_D^{-\lambda}e^{\gamma(1-\lambda)t}\lambda(B_RB_D\ell_{Rt})^{\rho_R}E_t^{\rho_R-1}\Big((A_Rk_{Rt})^{\rho_R}+(B_RB_DE_t\ell_{Rt})^{\rho_R}\Big)^{\frac{\lambda}{\rho_R}-1}.$$

By substitutions into (61), followed by equality (60), we have

$$\begin{aligned}\frac{\partial H_t}{\partial E_t}&=\mu_{Kt}(1-\alpha)+\mu_{Lt}(\gamma-d)\\ \implies \dot{\mu_{Lt}}-\delta\mu_{Lt}&=-\mu_{Kt}(1-\alpha)-\mu_{Lt}(\gamma-d)\\ \implies \dot{\mu_{Lt}}&=\mu_{Lt}(\delta-\gamma+d)-c_1e^{(\delta-r)t}(1-\alpha).\end{aligned}\tag{62}$$

The solution to this differential equation is

$$\mu_{Lt}=c_2e^{(\delta-\gamma+d)t}+\frac{c_1(1-\alpha)}{r-\gamma+d}e^{(\delta-r)t}.\tag{63}$$

The transversality condition on E implies that $c_2=0$, and hence that

$$\mu_{Lt}=\frac{c_1(1-\alpha)}{r-\gamma+d}e^{(\delta-r)t}=\frac{1-\alpha}{r-\gamma+d}\mu_{Kt}.\tag{64}$$

By (54) and (56),

$$k_{Dt}=B_DA_D^{\frac{\rho_D}{1-\rho_D}}E_t\ell_{Dt}.\tag{65}$$

Substituting this into (54), we have

$$\mu_{Kt}=A_D^{1-\eta}\Big(1+A_D^{\frac{\rho_D}{\rho_D-1}}\Big)^{\frac{1-\eta}{\rho_D}-1}k_{Dt}^{-\eta}\tag{66}$$

$$\implies k_{Dt}=k_{D0}e^{\frac{r-\delta}{\eta}t},\tag{67}$$

where

$$k_{D0}=c_1^{-\frac{1}{\eta}}A_D^{\frac{1-\eta}{\eta}}\Big(1+A_D^{\frac{\rho_D}{\rho_D-1}}\Big)^{\frac{1-\eta-\rho_D}{\rho_D\eta}}.\tag{68}$$

Now, from (55) and (58), we see that

$$k_{Rt}=\big(A_R/B_R\big)^{\frac{\rho_R}{1-\rho_R}}B_DE_t\ell_{Rt}.\tag{69}$$

Substituting this into (55), we have

$$\mu_{Kt}=\mu_{Lt}B_D^{-\lambda}e^{\gamma(1-\lambda)t}\lambda A_R^{\lambda}\Big(1+\big(A_R/B_R\big)^{\frac{\rho_R}{\rho_R-1}}\Big)^{\frac{\lambda}{\rho_R}-1}k_{Rt}^{\lambda-1}\tag{70}$$

$$\implies k_{Rt}=k_{R0}e^{\gamma t},\tag{71}$$

by (64), where

$$k_{R0}=\Big(\frac{\lambda(1-\alpha)}{r-\gamma+d}\Big)^{\frac{1}{1-\lambda}}\Big(\frac{A_R}{B_D}\Big)^{\frac{\lambda}{1-\lambda}}\Big(1+\big(A_R/B_R\big)^{\frac{\rho_R}{\rho_R-1}}\Big)^{\frac{\lambda-\rho_R}{\rho_R(1-\lambda)}}.\tag{72}$$

Interestingly, as we can see, k_{R0} does not depend on the resources at the movement's disposal. That is, though the right amount to spend on recruitment is proportional to labor productivity B_D and so grows at rate γ over time, it is determined entirely by the "calendar year", not at all by any property of the movement itself.

A rough intuition for this result is that, in a model with earning to give and hiring, labor and capital are ultimately highly substitutable, even if they are nominally gross complements in both direct work and recruitment. The optimal investment plan thus essentially involves "investing" in recruitment until its returns have fallen to the constant interest rate offered by financial investment.

Indeed, the optimal plan may even involve borrowing at r to invest in recruitment until the returns to doing so have fallen sufficiently, if we do not impose a no-borrowing side constraint. This is seen explicitly in (83); the capital stock may temporarily go negative.

From (51), we have

$$\dot{E}_t = (\gamma - d)E_t + B_D^{-\lambda} e^{\gamma(1-\lambda)t}\Big((A_R k_{Rt})^{\rho_R} + (B_R B_D E_t \ell_{Rt})^{\rho_R}\Big)^{\frac{\lambda}{\rho_R}}. \tag{73}$$

Thus, from (55),

$$\mu_{Kt} = \mu_{Lt}\lambda \frac{\dot{E}_t - (\gamma - d)E_t}{k_{Rt}} \frac{A_R^{\rho_R} k_{Rt}^{\rho_R}}{(A_R k_{Rt})^{\rho_R} + (B_R B_D E_t \ell_{Rt})^{\rho_R}} \tag{74}$$

$$\implies \dot{E}_t = c_E e^{\gamma t} + (\gamma - d)E_t, \tag{75}$$

where

$$\begin{aligned} c_E &\triangleq \frac{r - \gamma + d}{\lambda(1-\alpha)}\Big(1 + \big(A_R/B_R\big)^{\frac{\rho_R}{\rho_R - 1}}\Big) k_{R0} \\ &= \Big(\frac{A_R}{B_D}\frac{\lambda(1-\alpha)}{r - \gamma + d}\Big)^{\frac{\lambda}{1-\lambda}} \Big(1 + \big(A_R/B_R\big)^{\frac{\rho_R}{\rho_R-1}}\Big)^{\frac{\lambda(1-\rho_R)}{\rho_R(1-\lambda)}}. \end{aligned} \tag{76}$$

This differential equation gives us

$$E_t = \Big(L_0 - \frac{c_E}{d}\Big)e^{(\gamma-d)t} + \frac{c_E}{d}e^{\gamma t}. \tag{77}$$

Knowing k_D and E, ℓ_D can be calculated from (65):

$$\ell_{Dt} = k_{D0} A_D^{\frac{\rho_D}{\rho_D - 1}} \frac{1}{B_D} \frac{e^{\frac{r-\delta}{\eta}t}}{\big(L_0 - \frac{c_E}{d}\big)e^{(\gamma-d)t} + \frac{c_E}{d}e^{\gamma t}}. \tag{78}$$

Since $r > \gamma - d$, $c_E > 0$. Thus ℓ_D tends to 0 if $\gamma > (r - \delta)/\eta$, to ∞ in the (by assumption standard) case in which the inequality is reversed, and to

$$k_{D0} \frac{A_D^{\frac{\rho_D}{\rho_D - 1}}}{B_D} \frac{d}{c_E} \tag{79}$$

if the terms are equal.

From (55) and (58), we have

$$\ell_{Rt} = k_{R0}\Big(\frac{A_R}{B_R}\Big)^{\frac{\rho_R}{\rho_R-1}}\frac{1}{B_D}\frac{e^{\gamma t}}{\big(L_0-\frac{c_E}{d}\big)e^{(\gamma-d)t}+\frac{c_E}{d}e^{\gamma t}} \tag{80}$$

$$\implies \lim_{t\to\infty}\ell_{Rt} = k_{R0}\Big(\frac{A_R}{B_R}\Big)^{\frac{\rho_R}{\rho_R-1}}\frac{d}{B_D c_E}. \tag{81}$$

We can now obtain K_t from its law of motion:

$$\dot{K}_t = rK_t - k_{Dt} - k_{Rt} + B_D E_t(1-\ell_{Dt}-\ell_{Rt}-\alpha). \tag{82}$$

Substituting our expressions for k_{Dt}, k_{Rt}, E_t, ℓ_{Dt}, and ℓ_{Rt}, the only solution of this differential equation compatible with the transversality condition on K is

$$\begin{aligned} K_t &= k_{D0}\Big(1+A_D^{\frac{\rho_D}{\rho_D-1}}\Big)\frac{\eta}{r\eta-r+\delta}e^{\frac{r-\delta}{\eta}t} \\ &\quad+\bigg(k_{R0}\Big(1+\Big(\frac{A_R}{B_R}\Big)^{\frac{\rho_R}{\rho_R-1}}\Big)-B_D(1-\alpha)\frac{c_E}{d}\bigg)\frac{1}{r-\gamma}e^{\gamma t} \\ &\quad+B_D(1-\alpha)\Big(L_0-\frac{c_E}{d}\Big)\frac{\gamma-d}{\gamma-d-r}e^{(\gamma-d)t}. \end{aligned} \tag{83}$$

Finally, by setting the net present value of total expenditures equal to K_0 plus the net present value of total earnings, we can pin down c_1:

$$K_0 = \int_0^\infty e^{-rt}\big[k_{Dt}+k_{Rt}-B_D E_t(1-\ell_{Dt}-\ell_{Rt}-\alpha)\big]dt \tag{84}$$

$$\begin{aligned} &= \int_0^\infty \bigg[c_1^{-\frac{1}{\eta}}A_D^{\frac{1-\eta}{\eta}}\Big(1+A_D^{\frac{\rho_D}{\rho_D-1}}\Big)^{\frac{1-\eta-\rho_D}{\rho_D\eta}}e^{\frac{r-r\eta-\delta}{\eta}t} \\ &\quad+k_{R0}e^{(\gamma-r)t} \\ &\quad-B_D(1-\alpha)\Big(L_0-\frac{c_E}{d}\Big)e^{(\gamma-d-r)t} \\ &\quad-B_D(1-\alpha)\frac{c_E}{d}e^{(\gamma-r)t} \\ &\quad+c_1^{-\frac{1}{\eta}}A_D^{\frac{\rho_D(1-\eta)}{(\rho_D-1)\eta}}\Big(1+A_D^{\frac{\rho_D}{\rho_D-1}}\Big)^{\frac{1-\eta-\rho_D}{\rho_D\eta}}e^{\frac{r-r\eta-\delta}{\eta}t} \\ &\quad+\Big(\frac{A_R}{B_R}\Big)^{\frac{\rho_R}{\rho_R-1}}k_{R0}e^{(\gamma-r)t}\bigg]dt \end{aligned} \tag{85}$$

$$\implies c_1 = \bigg[K_0-\frac{k_{R0}}{r-\gamma}\Big(1+\Big(\frac{A_R}{B_R}\Big)^{\frac{\rho_R}{\rho_R-1}}\Big)+B_D(1-\alpha)\Big(\frac{L_0-c_E/d}{d+r-\gamma}+\frac{c_E}{d(r-\gamma)}\Big)\bigg]^{-\eta}$$

$$\cdot \left(\frac{r\eta - r + \delta}{\eta}\right)^{-\eta}\left(1 + A_D^{\frac{\rho_D}{1-\rho_D}}\right)^{\frac{(1-\eta)(1-\rho_D)}{\rho_D}} \tag{86}$$

This completes the solution.