Patience and Philanthropy

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This is a REWRITE IN PROGRESS of what I formerly called “Discounting for Patient Philanthropists”.

Abstract

I explore the implications of time preference heterogeneity for public good funding. I find that, across a variety of circumstances, low-time-preference funders (“patient philanthropists”) should often invest, rather than spend, the entirety of their resources for substantial lengths of time; and that the patient payoff to doing so, relative to that of spending impatiently, grows arbitrarily large as the patient philanthropist’s share of initial funding goes to zero. I also find that, when all preference heterogeneity across funders consists of time preference heterogeneity, fully efficient spending can be implemented in subgame-perfect equilibrium. Finally, I discuss some basic applications of these results to the timing of philanthropic spending on poverty alleviation efforts.

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1 Introduction

Agents must allocate their resources between spending and investment for future spending. Individuals, for instance, must decide how much of their income to consume and how much of it to save for retirement. Their decision will depend in part on whether they discount the utility from their future consumption, and if so, by how much. If individuals are assumed to discount this utility exponentially, the discount rate is typically estimated to be about 2% per year. Following this revealed time preference, it is standard practice for policymakers to discount future costs and benefits likewise (see e.g. US Council of Economic Advisers (2017)).

Some potential public projects require up-front costs for long-term benefits. Climate change mitigation, broadly speaking, is widely understood to be such a project. If a pure time preference rate anywhere close to 2% per year is used in setting climate policy, therefore, we may conclude that it is optimal to pursue only mild climate change mitigation efforts: a low carbon tax, for instance. Considering it unethical to let the impatience of the present generation impose such costs on future generations, Stern (2007) famously argues that (at least) climate policy should maintain “intergenerational equity”: that is, it should not reflect pure time preference. In place of a pure time discount rate of 2% per year, he proposes using a rate of 0.1% per year, intended to reflect only the annual risk of an exogenous catastrophe that renders climate change abatement efforts worthless. Stern’s recommendation, in effect, is to abandon time preference per se, but to act on the supposition that the life expectancy of our civilization is one thousand years.

This recommendation sparked a substantial literature on optimal policy by policymakers with lower rates of (quasi-)pure time preference than their constituents in general (typically 0.1%, following Stern), not just in the context of climate change. Optimal taxation problems, for example, are explored by Farhi and Werning (2007, 2010) in an overlapping generations context where individuals save insufficiently (from the patient social planner’s perspective) for their descendants, and by von Below (2012), Belfiori (2017), and Barrage (2018) in the context where present production confers both future costs and future benefits (from climate damage and capital accumulation respectively). An important lesson from this literature is that patient policymakers might like to invest for future spending, but that to avoid crowding out private investment, it is optimal for them instead to subsidize private investment and tax private consumption.
Not everyone agrees with the Stern recommendation. Many, such as Nordhaus (2007), defend the standard practice on the grounds that it is undemocratic for policymakers to use lower discount rates in setting public policy than individuals use in making their own intertemporal tradeoffs. Still, thirty-eight percent of economists who study discounting, according to a recent survey (Drupp et al. (2018)), and most moral philosophers (Broome (1994)), agree that, at least when setting policy, we ought to be patient.

Many philanthropists—that is, private actors seeking to contribute to the funding of public goods—endorse patience as well. Perhaps most notably, the Effective Altruism (EA) movement brands itself in part as a community of impartial, and by extension patient, philanthropists. Furthermore, the case for patience in a philanthropic context is presumably stronger than in a political one, since philanthropists are private actors pursuing (at least sometimes) ethical goals, and are not as beholden to potentially impatient constituents. Nevertheless, there appears to be essentially no academic discussion of the implications of patience for philanthropy. Though there are large existing literatures on dynamic public good provision and on time preference heterogeneity between policymakers and households, this paper is the first to consider the implications of time preference heterogeneity for dynamic public good provision.

This is particularly unfortunate because of the potentially large gap between the spending behavior of today’s patient philanthropists and patient-optimal philanthropic spending behavior. As noted above, according to standard economic assumptions, the patient should typically spend their resources more slowly than the impatient. A very low spending rate, however, would be a substantial departure from current common practice, even among philanthropists with expressed moral commitments to intergenerational equity. The EA community’s wealthiest members, for example, are currently Dustin Moskovitz and Cari Tuna, who plan to give away almost all of their roughly $11B net worth during their lifetimes (Matthews (2018)). Members of Giving What We Can, an EA-affiliated nonprofit that encourages individuals to pledge at least ten percent of their lifetime incomes to effective charities, have also collectively pledged an estimated $1.5 billion, mostly from smaller donors. The Giving What We Can pledge does not impose any particular spending schedule, but the organization recommends that its members give sooner (Giving What We Can (2013)). This tendency is not unanimous: a few have argued that investment returns exceed the rate at which doing good
is growing more costly (see e.g. Hanson (2013)), and some appear recently to be rethinking their old positions for giving now (see e.g. Vivalt (2019)). But advocating for less philanthropic spending and more long-term philanthropic investment is still far from mainstream in philanthropic circles, including the EA movement, and little effort has been put into its implementation.

Perhaps ironically, most large philanthropists outside the EA movement, who typically have made no explicit commitment to patience, already spend as slowly as the law permits foundations to spend—in the United States, that is 5% of their assets per year—and this behavior has historically met with criticism from those involved in EA (see e.g. Karnofsky (2007)). Most small donors outside EA, however, also appear to spend their charity budgets as soon as they earn them. This is difficult to measure precisely, but note that individual giving by Americans totaled $427.71B in 2018 (Giving USA (2019)), whereas contributions by Americans to donor-advised funds—tax-exempt vehicles in which donors without private foundations can invest funds for future giving—totaled $37.12B in 2018 (National Philanthropic Trust (2019)).

The most important conclusions for patient philanthropists are straightforward. Philanthropists face interesting strategic considerations that render their discounting problem different from that of typical policymakers and households. Nevertheless, they do not appear to be exceptions to the rule that the patient should spend slowly. The case for currently investing most or all “patient philanthropic resources” is therefore substantially stronger than it would seem from existing discussion. This conclusion would hold even if patient philanthropists were the only funders of the goods they intended to provide, and the presence of impatient funders further strengthens the case that patient philanthropists should invest. Finally, if the arguments presented here are broadly valid, the benefits to correct reasoning about discounting are large: patient philanthropists will often achieve their goals dramatically more effectively by spending patient-optimally than by spending impatient-optimally.

In short, if it can be demonstrated that patient philanthropists should spend more slowly, these conclusions could be highly relevant to the spending of at least tens of billions of dollars. An attempt to model the philanthropist’s intertemporal spending problem therefore appears to be worthwhile.

The structure of the paper is as follows.
In §2, I observe that the public good provision timing problem faced by a single funder, including a philanthropist with an unusually low discount rate, is analogous to a household’s basic consumption-smoothing problem. I also estimate the importance of the discount-rate-setting problem, from a patient perspective.

In §3, I consider a setting in which a public good has multiple funders with differing discount rates, so that patient philanthropists funding this good must worry about the extent to which their saving crowds out others’ saving. This work resembles existing work on private contributions to public goods (e.g. by Bergstrom et al. (1986)), in which contributors must worry about the extent to which their contributions crowd out others’. I address inter- rather than intra-temporal crowding out, however, and so assume that the net present value of the budget each party has allocated to the public good is fixed. Among other results, I find that when all preference heterogeneity across funders consists of time preference heterogeneity, fully efficient spending can be implemented in subgame-perfect equilibrium.

In §4, I consider a setting in which patient philanthropists can match contributions by their less patient counterparts, and I determine the matching schedule patient philanthropists should offer. This work closely resembles existing work on optimal taxation by policymakers with lower discount rates than their constituents. As discussed above, the relevant lesson from this literature is that patient policymakers will subsidize private investment and tax private consumption. Since I am addressing a patient philanthropist rather than a patient policymaker, however, I focus on the constraint that only subsidization, and not taxation, is feasible.

§5 explores the relationship between the conclusions from the earlier sections and existing discussion about the optimal timing of philanthropic spending, as produced by academics (primarily Andreoni (2018)) and by philanthropists outside academia (primarily those within the EA movement).

§6 concludes.

2 Single-funder model

2.1 Model

Let us begin with a model in which an agent is the sole provider of some good over an infinite horizon. Consider, for example, the case of a philanthropist
providing non-durable consumption goods for a penniless (but potentially long-lived) individual or lineage.

Let us denote the size of the agent’s budget at time \( t = 0 \) by \( B \). At each moment \( t \), we will assume that the flow utility \( u \) achieved by providing the good is an isoelastic function, with inverse elasticity of intertemporal substitution \( \eta > 0 \), of the rate \( X \geq 0 \) at which the agent spends. That is,

\[
    u(X(t)) = \begin{cases} 
    \frac{X(t)^{1-\eta} - 1}{1-\eta}, & \eta \neq 1; \\
    \ln(X(t)), & \eta = 1.
    \end{cases}
\]  

(1)

The agent faces a constant instantaneous real interest rate \( r \) and a constant instantaneous time preference rate \( \delta \). The latter might represent pure time preference, plus the risk of a catastrophe that brings the agent’s utility to zero forever after. (This could be the rate of “existential catastrophe”, i.e. the risk per unit time that the world ends or human civilization collapses. Less dramatically, in the case of a philanthropist that cares only about a beneficiary individual or lineage, it could represent this beneficiary’s mortality risk.) There does not appear to be a standard term for the quantity we are denoting \( \delta \), but we will call it the “time preference rate”, reserving the term “discount rate” for the discounting of marginal spending and the term “pure time preference rate” for time preference in a risk-free environment. We need not assume that \( r \) or \( \delta \) is positive.

The agent’s problem is then to choose the schedule of spending rates \( X(t) \) that maximizes

\[
    U = \int_{0}^{\infty} e^{-\delta t} u(X(t)) dt
\]

subject to the constraint

\[
    \int_{0}^{\infty} e^{-rt} X(t) dt \leq B.
\]

Proposition 1. Optimal individual spending schedule

Suppose an agent has isoelastic utility in spending parameterized by \( \eta \), a constant time preference rate \( \delta \), and a budget \( B \), and suppose she can invest her resources at a constant interest rate \( r \). Then the agent maximizes discounted utility by following spending schedule

\[
    X(t) = B \frac{r\eta - r + \delta}{\eta} e^{\frac{-\delta}{\eta} t}.
\]
Proof. See Appendix A.1.

Throughout this paper we will work in continuous time, roughly following the framework of Stinchcombe (2013). We will say that an optimal spending rate or equilibrium spending rate profile in continuous time is defined to be the limit as \( n \to \infty \) of a sequence of corresponding dynamic optimization problems or games in discrete time, with time increments for problem or game \( n \) given by the grid

\[
G_n \triangleq \left\{ \left[ \frac{k}{n!}, \frac{k+1}{n!} \right) \right\}, \quad k \in \mathbb{N}_{\geq 0}.
\]  

(4)

Infinitesimals are defined by sequences of real numbers which tend to zero, and are held to be strictly larger than zero but strictly less than any positive real number. In particular, the increment length is given by \( dt \triangleq \{1, \frac{1}{2}, \frac{1}{6}, \ldots\} > 0 \). Thus, for example, infinitesimal deviations from the spending schedule of Proposition 1, which cannot be obtained as the limit of optimal spending schedules over the discrete grid, are found to offer the agent a strictly lower payoff. For a simple illustration, consider the spending schedule

\[
\tilde{X}(t) = \begin{cases} 
X(t), & t \neq 1; \\
X(t)/2, & t = 1
\end{cases}
\]  

(5)

where \( X \) is defined as in Proposition 1. The payoff offered by \( \tilde{X} \) is \( e^{-\delta}(u(X(1)) - u(X(1)/2))dt \) less than the payoff offered by \( X \).

As we can see, grid sequence \( G \) approximates continuous time, in the sense that the measure of its partition elements at \( n \) can be bounded above by a value \((1/\n!)\); in fact all of \( G_n \)'s partition elements are precisely of this measure) which tends to zero as \( n \) increases. Note that alternative grid sequences which also approximate continuous time can in some cases produce different behavior on measure-zero sets of times for the same discrete-time optimization problems or games. The definition of a continuous-time optimization problem or game is thus technically sensitive to the grid specification, though in the cases we discuss, optimal or equilibrium behavior and payoffs will differ only infinitesimally from one grid sequence to another. The grid sequence \( \{G_n\} \) we have chosen is designed to ensure that each rational number partitions all grids \( G_{n'}(n' \geq n) \) for some finite \( n \). This will in some ways simplify the analysis of continuous-time behavior.
2.2 Discussion

The above model is motivated in this paper by the scenario in which a philanthropist is the sole provider of some good. So far, it is equivalent to an infinite-horizon consumption-smoothing model under certainty, assuming either (a) no future outside income or (b) complete capital markets. (Note that the assumption of complete capital markets renders this problem the same as the problem one faces with no outside income. Given certainty and complete markets, someone with future income can borrow against her entire income stream, and $B$ can represent current assets plus the present value of future income.) Nevertheless, given the centrality of the underlying relationship described above to the analysis of patient philanthropy below, let us now take a moment to note three of its relevant features.

First, and most importantly: In the context of a simple consumption-smoothing model, the optimal spending rate is highly sensitive to the discount rate. The patient, that is, should spend slowly. As we can see from (8) at $t = 0$, it is always optimal to spend at proportional rate $\frac{\eta r - r + \delta}{\eta}$. In particular, if $\eta = 1$, the spending rate should equal $\delta$. For instance, philanthropists who are funding idiosyncratic projects with no other present or future funders, who discount future impacts at 0.1% per year, and who are confident that the world (or their philanthropic projects) will not soon be brought to an end, should spend only 0.1% of their budgets per year.

Second: Whether outflows are increasing, constant, or decreasing in time depends on whether $r - \delta$ is greater than, equal to, or less than zero. Furthermore, if $r - \delta > 0$, the rate of increase in spending is also increasing with time, and if $r - \delta < 0$, $\lim_{t \to \infty} x(t) = 0$. It follows that we have no edge cases in which one’s assets should be expected to grow or shrink asymptotically to a positive size. The $r = \delta$ steady state is unstable. If a fund should grow, it should not stop growing, even once it has grown very large.

Third: Unless $\delta > r(1 - \eta)$, (6) we appear to be led to the conclusion that it is always preferable to invest than to spend. This is, in other words, the condition necessary to avoid the Koopmans (1967) “paradox of the indefinitely postponed splurge”. For the purposes of this paper, rather than broach the subject of infinite ethics, we will assume that this condition holds. Note that it does whenever $\eta > 1$, $r > 0$, and $\delta \geq 0$. That is, under the common assumptions that $r > 0$ and $\eta > 1$, agents can be fully patient without entering paradoxical territory.
2.3 The value of patience

As noted in §1, philanthropists with expressed ethical commitments to inter-generational equity sometimes spend quickly, as might appear to be optimal given a positive and substantial rate of pure time preference but not otherwise. A cynical economist’s interpretation of this phenomenon would be that such philanthropists know what they are doing, and that their behavior reveals that do have pure time preference; indeed, Alexander (2013) reports Robin Hanson taking this position. To the extent that their impatient spending behavior is simply mistaken, however, we can estimate the size of this error. That is, we can estimate true patient philanthropists’ willingness to pay (even if they do not yet know it) to move from the impatient-optimal to the patient-optimal spending rate.\footnote{Having now personally discussed the issue with many philanthropists, including some with assets in the ten figures, I can anecdotally report that they have not thought about discounting in this way, and do value work along these lines. Indeed, some are indirectly funding this research.}

Furthermore, large philanthropists face a choice between holding their capital in a foundation and holding it privately or in a trust. In the United States, contributions to foundations are tax-exempt, as are the capital gains their assets earn. Foundations must disburse at least 5% of their assets per year, however, effectively requiring them to act impatiently. Trusts are not tax-exempt but are not subject to such a requirement. An analysis like the following can inform large patient philanthropists about how significant the tax advantage to a foundation must be to justify this loss of freedom over the implicit choice of time preference rate.

To begin, let us determine the patient payoff to spending according to some time preference rate $\tilde{\delta} \geq \delta$.

The patient payoff to spending according to $\tilde{\delta}$ can be found by using $\tilde{\delta}$ in the expression for Proposition 1 to get the $\tilde{\delta}$-optimal spending schedule. Then, substitute this schedule as $x(t)$ into (2) to get

$$\int_0^\infty e^{-\delta t} u\left(B \frac{r\eta - r + \tilde{\delta}}{\eta} e^{\frac{r - \tilde{\delta} t}{\eta}}\right) dt.$$  \hfill (7)

Observe that this will only be defined if

$$\eta \leq 1 \text{ or } \tilde{\delta} < r + \delta \frac{\eta}{\eta - 1}. \hfill (8)$$
If \( \eta > 1 \) and \( \tilde{\delta} \) is too high, the \( \tilde{\delta} \)-optimal plan may push the spending rate to 0 quickly enough that, though this produces finite \( \tilde{\delta} \)-discounted disutility, it produces infinite \( \delta \)-discounted disutility. Note that \( \tilde{\delta} < r + \delta \) is sufficient to avoid this condition.

**Proposition 2. Payoff to spending according a given time preference rate**

Suppose an agent satisfies the conditions of Proposition 1. Then her payoff to following the \( \tilde{\delta} \)-optimal spending schedule, for some \( \tilde{\delta} \geq \delta \), is

\[
U_\delta(B, \tilde{\delta}) = \begin{cases} 
\frac{B^{1-\eta/(r \eta - r + \tilde{\delta})} - 1}{(1-\eta)(r \eta - r - (r - \tilde{\delta})(1-\eta))}, & \eta \neq 1; \\
\frac{\delta \ln(B \tilde{\delta}) + r - \tilde{\delta}}{\delta^2}, & \eta = 1.
\end{cases}
\]

**Proof.** Integrate (7) subject to (8). \( \square \)

We can now calculate how much of her budget a patient agent should be willing to give up to move from the \( \tilde{\delta} \)-optimal to the \( \delta \)-optimal spending schedule.

**Proposition 3. WTP for patience**

Suppose an agent satisfies the conditions of Proposition 1. Then, in order to spend her resources as would be optimal given time preference rate \( \delta \) as opposed to \( \tilde{\delta} \geq \delta \), she is willing to give up the following fraction of her budget:

\[
1 - \frac{r \eta - r + \tilde{\delta}}{r \eta - r + \tilde{\delta}} \left( \frac{r \eta - r + \tilde{\delta}}{r \eta - r + \tilde{\delta} - (\tilde{\delta} - \delta) \eta} \right)^{1/\eta}, \eta \neq 1;
\]

\[
1 - \frac{\tilde{\delta} e^{1-\tilde{\delta}}}{\delta}, \eta = 1.
\]

**Proof.** Using the payoff expressions from Proposition 2, set \( U_\delta(B, \tilde{\delta}) = U_\delta((1 - w)B, \delta) \) and solve for \( w \). \( \square \)

This fraction is decreasing in \( \eta \), as shown numerically in the online appendix accompanying this paper.

Concretely, suppose \( r = 5\% \). Then the value achieved by spending according to time preference rate \( \tilde{\delta} = 2\% \), by the lights of time preference rate \( \delta \), is equal to the value achieved by giving up the following budget-fractions but spending the remaining budget according to time preference rate \( \delta \):
Recall that 2\% is a common rough estimate of the time preference rate that most agents employ.

As we can see, a patient agent can err substantially by spending as would be optimal given a more typical time preference rate. That is, implicitly spending according to spending rate \( \delta = 2\% \) is a mistake she should be willing to give up a substantial part of her budget to avoid. Furthermore, this willingness to pay is highly sensitive to the values of \( \eta \) and \( \delta \). It is most extreme for low values of \( \eta \) and \( \delta \): in this case it is almost her entire budget. Even when \( \eta = 2 \) and \( \delta = 0.5\% \), however, spending impatiently is tantamount to a loss of about 7\% of her resources.

### 3 Free-riding and crowding out

#### 3.1 Motivation and framework

The model above allows us to determine the optimal spending policy regarding the provision of a good for which there is only one purchaser. It applies, for instance, to the schedule on which an individual should allocate her private spending, or on which a philanthropist only interested in funding an esoteric project should allocate his spending on that project. When one is a philanthropist providing a public good to which others contribute, however, one must consider the ways in which one’s own funding affects the behavior of the good’s other funders. In particular, when one is a patient philanthropist, one must remember that investment for future spending can induce less patient funders to spend more quickly.

As we will see, intertemporal free-riding and crowd-out concerns can motivate substantially different—and generally even “more patient”—behavior from a patient philanthropist than is optimal in the single-funder context. In

<table>
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<th>( \eta )</th>
<th>( \delta = 0.1% )</th>
<th>( \delta = 0.5% )</th>
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<tbody>
<tr>
<td>0.99</td>
<td>( 1 - 4.0 \times 10^{-13} )</td>
<td>0.84</td>
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<tr>
<td>1</td>
<td>( 1 - 1.1 \times 10^{-7} )</td>
<td>0.80</td>
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<tr>
<td>1.01</td>
<td>( 1 - 1.8 \times 10^{-5} )</td>
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<td>1.25</td>
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In particular, a patient philanthropist often does best in the presence of impatient funders to invest \textit{all} his resources, for some period, and then to spend on an exponential schedule resembling the single-funder schedule determined above.

Throughout the results below, we will posit a single impatient party and a single patient party. We will denote party \( i \)'s budget at time \( t \) by \( B_i(t) \), the total budget by \( B(t) \equiv B_I(t) + B_P(t) \), and \( i \)'s budget proportion by \( b_i(t) \equiv B_i(t)/B(t) \), for \( i \in \{I,P\} \). Terms without time arguments \((B_i,B,b_i)\) will denote their values at \( t = 0 \). Finally, we will denote party \( i \)'s time preference rate by \( \delta_i \), for \( i \in \{I,P\} \). We will assume that the time preference rates satisfy conditions (6) and (8), with \( \delta_P \) as \( \delta \) and \( \delta_I \) as \( \tilde{\delta} \).

We will introduce a more precise framework when necessary, at the beginning of §3.3. For now, let us omit some technicalities and say only that at every moment \( t \geq 0 \), the players observe the spending history \( \{(X_I(s),X_P(s))\}_{s<t} \) and independently choose spending rates \( X_I(t) \) and \( X_P(t) \) respectively. The patient utility function is then given by

\[
U_P = \int_0^\infty e^{-\delta_P t} u(X_I(t) + X_P(t)) dt,
\]

where \( u(\cdot) \) is an isoelastic function parametrized by \( \eta \), as before.

Throughout the following sections, it will be helpful to define the following terms:

\[
\alpha_I \equiv \frac{r\eta - r + \delta_I}{\eta}, \\
\alpha_P \equiv \frac{r\eta - r + \delta_P}{\eta}, \\
\gamma = \left( \frac{\alpha_P + \delta_I - \delta_P}{\alpha_I} \right)^{\frac{1}{1-\eta}}.
\]

Note that all three terms are positive under all admissible parameters, except \( \gamma \), which is undefined when \( \eta = 1 \).

\[^2\text{If} \ P \text{ rejects pure time preference altogether,} \ \delta_I - \delta_P \text{ equals the portion of the impatient time preference rate consisting of pure time preference, as distinct from Stern-style discounting for risk of civilizational collapse.}\]

\[^3\text{For a model in which actors with different discount rates jointly set spending rates over time, see Millner and Heal (2018).}\]
3.2 Response to a warm-glow impatient funder

Following Andreoni (1990), let us define two forms that the impatient funder’s utility function might take.

**Definition 1.** The impatient funder is **altruistic** if her utility function is given by

\[ U_I = \int_0^\infty e^{-\delta t} u(X_I(t) + X_P(t)) dt. \]

**Definition 2.** The impatient funder is **warm-glow** if her utility function is given by

\[ U_I = \int_0^\infty e^{-\delta t} u(X_I(t)) dt, \]

with \( X_I(t) \), rather than \( X_I(t) + X_P(t) \), as the argument of her flow utility function.

As one might intuit, and as we will later see, the presence of the patient funder may motivate an altruistic impatient funder to spend her budget more quickly, in anticipation of the patient funder’s future spending. An assumption of this sort is plausible if the impatient party is the patient philanthropist’s beneficiary—that is, if the public good to which they are both contributing is the impatient party’s consumption—and especially so if the beneficiary can borrow against the philanthropist’s future transfers.

If \( \delta_P \) is sufficiently low, however, the patient party may plan to invest most or all his resources for centuries before spending them for the benefit of the current generation’s distant descendants. Borrowing against this far-future income is typically impossible, and we might consider it implausible that the current generation would alter its spending and bequest decisions on the basis of such a distant prospect.

Let us therefore begin by determining patient-optimal spending behavior in the presence of a warm-glow impatient funder.

**Proposition 4. Patient spending given a warm-glow impatient funder**

In the presence of a warm-glow impatient funder, the patient funder does best to follow spending schedule

\[ X_P(t) = \begin{cases} 0, & t < t^*; \\ \left( B_I e^{(r-a_I)t^*} + B_P e^{r(t-t^*)} \right) \alpha_P e^{(r-a_P)(t-t^*)} - B_I \alpha_I e^{(r-a_I)t}, & t \geq t^*, \end{cases} \]
where
\[
t^* = \max \left( 0, \ln \left( \frac{B_I \alpha_I - \alpha_P}{B_P \alpha_P} \right) / \alpha_I \right).
\]

Proof. See Appendix A.2. \qed

As we can see, the patient party does best to invest all his resources as long as his share of total resources is sufficiently low: in particular, as long as
\[
b_P(t) < \frac{\alpha_I - \alpha_P}{\alpha_I}.
\]
(10)
The intuition is straightforward. If the impatient party controls a large enough share of total resources, the impatient-optimal spending rate at which to spend her own budget may be higher not just than the patient-optimal rate at which to spend her budget, but than the patient-optimal rate at which to spend the collective budget. If so, any spending by the patient party would, on his view, increase the extent to which they are collectively over-spending. He should only begin spending once the impatient party’s share of the collective budget has shrunk enough that even spending it impatient-optimally constitutes underspending the collective budget, from the patient perspective.

3.3 Interaction with an altruistic impatient funder

If the impatient funder is altruistic, the actors’ spending problems take the form of a dynamic game. There is already a substantial literature on dynamic public goods contribution games, but none of it yet appears to have considered the implications of differences in time preference. The framework used here is designed to introduce such differences. In fact it isolates the effects of differences in time preference by positing that they are the funders’ only preference differences; funders do not have the opportunity to spend on private goods which only they value, as they are typically assumed to have. We will therefore hold fixed the size of the budget each party contributes to the public good.

Before exploring the dynamic game in question, however, let us consider two closely related static games.

First, let us suppose that each player commits to a spending schedule in advance, but not to a history-dependent spending policy.
Proposition 5. Existence and uniqueness of Nash equilibrium in the simultaneous-move game
Suppose that, at $t = 0$, each funder $i$ sets the entire spending schedule $X_i(t)$ simultaneously. This game has a Nash equilibrium in which

$$X_i^*(t) = \begin{cases} (B_I \alpha + B_P \alpha_P) e^{(r-\alpha_I) t}, & t < t^*; \\ 0, & t \geq t^* \end{cases}$$

and

$$X_P^*(t) = \begin{cases} 0, & t < t^*; \\ B_P \alpha_P \left(1 + \frac{B_I \alpha_I}{B_P \alpha_P} \right)^{\frac{\alpha_P}{\alpha_I}} e^{(r-\alpha_P) t}, & t \geq t^*, \end{cases}$$

where

$$t^* = \ln \left(1 + \frac{B_I \alpha_I}{B_P \alpha_P} \right) / \alpha_I.$$ 

This Nash equilibrium is the game’s unique Nash equilibrium.\(^4\)

Proof. See Appendix A.3. \(\square\)

This is essentially a special case of the static public good contribution game analyzed by Bergstrom et al. (1986), but with a continuum of public goods: spending at each $t$, for $t \in [0, \infty)$. As in the analogous Bergstrom et al. case, we here find that each good is provided by exactly one funder, and a funder always provides a good when he is also providing a good for which he cares relatively less. That is, since $P$ cares relatively more than $I$ about spending at $t$ the later $t$ is, there is a threshold time $t^*$ such that $I$ is the sole funder before $t^*$ and $P$ is the sole funder after.

Furthermore, the spending rate is continuous at $t^*$; $\lim_{t \uparrow t^*} X_i^*(t) = X_P^*(t^*)$. If the spending rate rose discontinuously at $t^*$, $P$ would do better to reallocate some spending from $t^* + \epsilon$ to $t^* - \epsilon$ for some sufficiently small

\(^4\)Total funding at $t^*$ itself, i.e. $X_i^*(t^*) + X_P^*(t^*)$, must equal the expression for $X_P^*(t^*)$ above. Strictly speaking, however, we can only guarantee that $P$ covers the entirety of this funding if $B_I$, $B_P$, $\alpha_I$, and $\alpha_P$ are such that $t^*$ is rational. If $t^*$ is irrational, our choice of grid sequence (see §2.1) may require funding at $t^*$ to come in part from $I$, or in pathological cases may even leave the funding split at $t^*$ undefined, without a modification to the choice of grid sequence or a “tie-breaking” modification to at least one agent’s utility function. We will encounter a similar complication in Propositions 6 and 7, which likewise feature “regime-switching” times $t^*$.

For simplicity of exposition, however, we will assume here and henceforth that $P$ covers spending at $t^*$.
\( \epsilon > 0 \). Likewise, if the spending rate fell discontinuously, I would do better to reallocate marginal spending forward.

Next, let us suppose that I is the “Stackelberg leader”.

**Proposition 6. Existence and uniqueness of subgame-perfect equilibrium in the Stackelberg game**

Suppose that, at \( t = 0 \), I sets spending schedule \( X_I(t) \), and \( P \) sets spending schedule \( X_P(t) \) in response. This game has a subgame-perfect equilibrium across the two periods, which induces spending schedules

\[
X_I^*(t) = \begin{cases} 
B_I \frac{Z}{\alpha I} \alpha_I e^{(r-\alpha_I)t}, & t < t^*; \\
0, & t \geq t^* 
\end{cases}
\]

and

\[
X_P^*(t) = \begin{cases} 
0, & t < t^*; \\
B_P \frac{\alpha_P}{\alpha_I} \alpha_P e^{(r-\alpha_P)t}, & t \geq t^* 
\end{cases}
\]

respectively, where

\[
t^* = \ln(Z) / \alpha_I
\]

and

\[
Z \triangleq \begin{cases} 
1 + \frac{B_I \alpha_I}{B_P \alpha_P} \gamma, & \eta \neq 1; \\
1 + \frac{B_I \delta_I}{B_P \delta_P} e^{\frac{\delta_P}{\eta} - 1}, & \eta = 1.
\end{cases}
\]

**Proof.** See Appendix A.4. \( \square \)

Since \( \gamma < 1 \) and \( e^{\frac{\delta_P}{\eta} - 1} < 1 \), the regime-switching time \( t^* \) occurs earlier in the Stackelberg case than in the case where the funders set their spending plans simultaneously. Furthermore, recall that in the simultaneous-move case, spending is continuous at \( t^* \). Here, the spending rate falls discontinuously at \( t^* \), as \( I \) allocates budget \( B_I \) over a shorter time interval and \( P \) stretches \( B_P \) over an infinite horizon beginning earlier.

We will now explore the funders’ interaction in a dynamic setting.

To formalize this game, we will define a complete (spending) history as an assignment of a spending rate \( X_{i,t} \) to each player \( i \) for each \( t \geq 0 \). It will be denoted by

\[
X \triangleq \{(X_{I,t}, X_{P,t})\}_{t=0}^{\infty}. \tag{11}
\]
An open partial history, i.e. one truncated just before some \( t \), will be denoted \( X_{|t} \). In a slight abuse of notation, we will some denote total spending at \( t \) by \( X_t \).

A feasible history \( X \) is one whose spending schedules are integrable and feasible for each player: that is, one in which
\[
\int_0^\infty e^{-rt} X_{i,t} dt \leq B_i \tag{12}
\]
for each \( i \). The set of all feasible histories will be denoted \( \mathbb{X} \).

We will denote the set of decision nodes by \( \mathbb{D} \). Note that this is precisely the set of open partial feasible histories: \( \mathbb{D} \triangleq \{ X_t \} : X \in \mathbb{X}, t \geq 0 \).

A strategy \( \sigma_i \) for player \( i \) is a function from nodes to spending rates, i.e.
\[
\sigma_i : \mathbb{D} \to \mathbb{R}_{\geq 0}. \tag{13}
\]
Player \( i \)’s strategy set will be denoted \( \Sigma_i \), strategy profiles will be denoted \( \sigma \triangleq (\sigma_I, \sigma_P) \), and the set of strategy profiles will be denoted \( \Sigma \triangleq \Sigma_I \times \Sigma_P \).

We will denote \( i \)’s budget at time \( t \) given feasible history \( X \)—i.e. at node \( X_{|t} \)—by \( B_i(X_{|t}) \). Likewise, the total budget will be denoted \( B(X_{|t}) \) and budget proportions \( b_i(X_{|t}) \).

We will denote by \( X(\sigma) \) the history induced by strategy profile \( \sigma \). Given \( X_{|t} \in \mathbb{D} \), if the players subsequently adopt strategy profile \( \sigma \), we will denote the resulting history by \( X(X_{|t}, \sigma) \).

Because the utility function we are working with is isoelastic and thus homothetic, spending at all times is proportional to \( B \), holding \( b_I \) and \( b_P \) constant. Instead of working with absolute spending \( X \), therefore, it will sometimes be useful to work with normalized spending \( x \), where \( x_t \triangleq X_t/B \).

A strategy profile \( \sigma^* \) is a subgame perfect equilibrium if, for all \( X_{|t} \in \mathbb{D} \),
\[
\int_t^\infty e^{-\delta t} u(X_t(\sigma^*)) dt \geq \int_t^\infty e^{-\delta t} u(X_t(X_{|t}, (\sigma_i, \sigma^*_j))) dt \quad \forall \sigma_i \in \Sigma_i \tag{13}
\]
for both players \( i \) (where \( j \) denotes the other player).
Definition 3. A **defection profile** of a dynamic public good contribution game among funders with time preferences $\delta_I > \delta_P$ is a strategy profile $\sigma^D$ in which

$$\int_0^t e^{-rs}X_{I,s}(\sigma^D)ds < B_I \iff \sigma^*_P(X|_t) = 0. \quad (14)$$

In other words, a defection profile $\sigma^D$ is one which maintains a history $X(\sigma^D)$ in which the patient party does not fund the public good until the impatient party has disbursed all her resources—i.e. until

$$t^* \triangleq \min\{t : B_I(X|_t(\sigma^*)) = 0\}. \quad (15)$$

As we will now see, the game has infinitely many subgame-perfect equilibria, but exactly one “defection equilibrium”. That is, one subgame-perfect equilibrium is a defection profile.

**Proposition 7. Existence and uniqueness of defection equilibrium**

In a two-player dynamic public good contribution game where flow utility is isoelastic in collective spending and the players have different time preference rates $\delta_I > \delta_P$, there is a unique “defection equilibrium” $\sigma^D$. It induces spending rates $X_{i,t}(\sigma^D) = X^D_{i,t}$ for each $i$, where the $X^D_{i,t}$ are defined as in Proposition 6.

**Proof.** See Appendix A.5. \qed

Let us call

$$X^D_{i,t} \triangleq X_{I,t}(\sigma^D) + X_{P,t}(\sigma^D) \quad (16)$$

the “defection schedule”. Like the Stackelberg schedule, it follows an impatient-optimal spending schedule for $t \in [0, t^*)$ and a patient-optimal spending schedule for $t \in [t^*, \infty)$. Because it is equal to the Stackelberg schedule, we know that $I$ strictly prefers it to the Nash schedule, which is also feasible for $I$ in the Stackelberg setting. By analogous reasoning it is strictly dispreferred to the Nash schedule by $P$, who would prefer to slow $I$’s spending with a later, rather than an earlier, regime-switching time $t^*$.

In effect, a public good contribution game between funders whose preference-differences consist of differences in time preference, the impatient funder has a first-mover advantage, allowing her to do better than her counterpart in a static public good contribution game between funders with
preference-differences over a continuum of single-period goods.

Let us call $x^D(b_I)$ the normalized defection schedule, interpreted as a function of $b_I$. As we can see from the formulas above, the homotheticity of $u$ ensures that the defection schedule is indeed fully characterized, with respect to the budget sizes, by $B$ and $b_I$.

Except in the trivial cases where $b_I = 1$ or 0 (so that $t^* = 0$ or $\infty$, respectively), $X^D$ is inefficient. The impatient party is indifferent regarding marginal reallocations of resources from $t_1 < t^*$ to $t_2 \in (t_1, t^*]$, whereas the patient party strictly prefers them. Likewise the patient party is indifferent regarding marginal reallocations of resources from $s_2 > t^*$ to $s_1 \in (t^*, s_2)$, whereas the impatient party strictly prefers them. If the parties could contract, therefore, they could achieve a Pareto improvement by shifting spending toward $t^*$ from both sides.

The following proposition shows, however, that an enforceable contract is not necessary to achieve efficiency.

**Proposition 8. Efficient equilibria**

A spending schedule is efficient iff it maximizes discounted utility using declining time preference rate

$$\delta(t) = \frac{a\delta_I e^{-\delta_I t} + (1 - a)\delta_P e^{-\delta_P t}}{ae^{-\delta_I t} + (1 - a)e^{-\delta_P t}}$$

for some $a \in [0, 1]$. Furthermore, every efficient Pareto improvement to the defection schedule can be obtained in a subgame perfect equilibrium.

**Proof.** See Appendix A.6. \hfill \Box

As we can see, an efficient spending schedule uses a time preference rate that declines from $a\delta_I + (1-a)\delta_P$ at $t = 0$ to $\delta_P$ as $t \to \infty$, as long as $a < 1$. In doing so, it naturally follows the same path as the declining discount rate that is optimal under discount rate uncertainty, for an agent placing probability $a$ on the validity of discount rate $\delta_I$ and probability $1 - a$ on that of $\delta_P$, as shown by Weitzman (1998). Furthermore, such a spending schedule is obtainable in equilibrium, without the need for explicit contracting between the parties.
3.4 The value of patience

As in §2.3, let us now determine “how much” a patient philanthropist errs by spending impatiently. This question will only be well-defined when inequality (8) is met, with $\delta = \delta_P$ and $\tilde{\delta} = \delta_I$; if $\delta_I$ is too high relative to $\delta_P$, spending impatiently can accrue infinite disutility from a patient perspective. Throughout this section, therefore, we will assume that this inequality holds.

From Proposition 2 at $\delta = \delta_P$, $\tilde{\delta} = \delta_I$, and $B = B_I + B_P$ (still denoted $B$), we have $P$’s payoff from spending impatiently. As in that context, let us denote this payoff $U_{\delta_P}(B, \delta_I)$. We will now determine $P$’s payoff to spending patiently in the presence of warm-glow and altruistic impatient funders respectively.

**Proposition 9. Payoff to patience given a warm-glow impatient funder**

In the presence of a warm-glow impatient funder, a patient funder spending optimally attains payoff

$$\frac{B^{1-\eta}}{1-\eta} \frac{\alpha_{I}^{-\eta}}{\delta_{P}^{-\eta}} - \frac{1}{\delta_{P}(1-\eta)}, \quad \eta \neq 1;$$

$$\frac{\ln(B\delta_P) - 1}{\delta_P} + \frac{r}{\delta_P^2}, \quad \eta = 1$$

if $b_P \geq (\alpha_I - \alpha_P)/\alpha_I$, and

$$\frac{B^{1-\eta}_{I}}{1-\eta} \alpha_{I}^{1-\eta} \left[ \left( \frac{1}{\delta_{I} - \delta_{P} - \alpha_{I}} + \frac{1}{\alpha_{P}} \right) \left( \frac{B_{I} \alpha_{I} - \alpha_{P}}{B_{P}} \frac{\delta_{I} - \delta_{P} - \alpha_{I}}{\alpha_{I}} \right) \right] - \frac{1}{\delta_{P}(1-\eta)}, \quad \eta \neq 1;$$

$$\frac{1}{b_{P}} \left[ \ln(B_{I}\delta_{I}) + \frac{r}{\delta_{P}} \left( \frac{B_{I}}{B_{P}} \right) + \left( \frac{b_{P}}{B_{P}} \right) \frac{\delta_{I}}{\delta_{P}} \frac{\delta_{I} - \delta_{P}}{\alpha_{I}} \right], \quad \eta = 1$$

if $b_P < (\alpha_I - \alpha_P)/\alpha_I$.

**Proof.** From Proposition 4, we know that, given a warm-glow impatient funder, $P$ is able to implement patient-optimal spending of the collective budget as long as $b_P \geq (\alpha_I - \alpha_P)/\alpha_I$. If this inequality is met, therefore, the payoff to spending patiently is the expression from Proposition 2 for $U_{\delta_P}(B, \delta_P)$.

If this inequality is not met, simply integrate $\delta_P$-discounted utility given the spending rates from Proposition 4. That is, calculate

$$\int_{0}^{\infty} e^{-\delta_{P}t} (B_{I}\alpha_{I}e^{(r-\alpha_{I})t}) dt + \int_{t_{*}}^{\infty} e^{-\delta_{P}t} u((B_{I}e^{(r-\alpha_{I})t_{*}} + B_{P}e^{rt})\alpha_{I}e^{(r-\alpha_{P})(t-t_{*})}) dt,$$

where $t_{*} = \ln \left( \frac{B_{I} \alpha_{I} - \alpha_{P}}{B_{P} \alpha_{P}} \right)/\alpha_I > 0.$
In the presence of an altruistic impatient funder, we have seen that there are multiple equilibria, offering the parties different payoffs. Without some equilibrium selection argument, therefore, the payoff to each party is indeterminate. Because of the defection equilibrium’s simplicity and Markov perfection, we will take the defection equilibrium payoffs to be a natural lower bound for each party.

**Proposition 10. Lower bound on payoff to patience given an altruistic impatient funder**

In the presence of an altruistic impatient funder, if the funders engage in the defection equilibrium or a Pareto-superior equilibrium, a patient funder attains a payoff of at least

\[
\frac{B P^{1-\eta}}{1-\eta} \frac{\alpha P^{1-\eta}}{\gamma (\delta I - \delta P - \alpha I)} \left[ (\gamma (\delta I - \delta P - \alpha I) + \alpha P) Z^{-\frac{\alpha P}{\delta P}} - \alpha P Z^{1-\eta} \right] - \frac{1}{\delta P (1-\eta)}, \eta \neq 1; \\
\frac{1}{\delta P} \left[ (\delta I - \delta P)^{2} Z^{\frac{1}{1-\eta}} + \ln (B P^P Z) + \frac{\delta I - \delta P}{\delta P} + \frac{r-\delta P}{\delta P} \right], \eta = 1,
\]

where \( Z \) is defined as in Proposition 6.

**Proof.** Integrate \( \delta P \)-discounted utility given the spending rates from Proposition 7. That is, calculate

\[
\int_{0}^{t^*} e^{-\delta P t} u (B I Z^{-\frac{1}{1-\eta}} e^{(r-\alpha I) t}) dt + \int_{t^*}^{\infty} e^{-\delta P t} u (B P Z^{\frac{1}{\alpha P}} e^{(r-\alpha P) t}) dt.
\]

\( \Box \)

**Proposition 11. WTP for patience given a warm-glow impatient funder**

Given a warm-glow impatient funder, a patient funder’s willingness to pay for patience, as a proportion of his budget, equals

\[
\left( 1 - \frac{\alpha I}{\alpha P} \left( \frac{\alpha P}{\alpha I + \delta P - \delta I} \right)^{1-\eta} \right) / b P, \eta \neq 1; \\
\left( 1 - \frac{\delta I}{\delta P} e^{1-\frac{\delta I}{\delta P}} \right) / b P, \eta = 1
\]

if \( b P \geq (\alpha I - \alpha P) / \alpha I \), and

\[
1 - \frac{1 - b P}{b P} \left( \frac{1}{1-b P} \right)^{1-\eta} - \frac{\alpha I^{1-\eta}}{\alpha I + \delta P - \delta I} \left( \frac{\alpha I - \alpha P}{\alpha P} \right)^{1-\eta}, \eta \neq 1; \\
1 - \frac{1 - b P}{b P} \left( \ln \left( \frac{1}{1 - b P} \right) \right)^{\frac{\delta I}{\delta P}} \left( \frac{\delta I - \delta P}{\delta P} \right)^{\frac{\delta I - \delta P}{\delta P}}, \eta = 1
\]
if $b_p < (\alpha_I - \alpha_P)/\alpha_I$. This is bounded below by the patient funder’s willingness to pay for patience when the patient funder is the only funder (the case of Proposition 3), which it approaches as $b_p \to 1$; is strictly decreasing in $b_p$; and approaches 1 as $b_p \to 0$.

**Proof.** From Proposition 3, we have $P$’s “willingness to pay for patience” as a fraction of the collective budget. Recall that if $b_p \geq (\alpha_I - \alpha_P)/\alpha_I$, $P$ can, by acting patiently, implement the patient–as opposed to the impatient–optimal spending schedule for the collective budget. In this case, therefore, $P$’s willingness to pay for patience as a fraction of the collective budget is the expression from Proposition 3, with $\delta_P$ as $\delta$ and $\delta_I$ as $\bar{\delta}$, so his willingness to pay as a fraction of $B_P$ is simply this expression from Proposition 3 divided by $b_P$.

In this case it is clear that the WTP is strictly decreasing in $b_P$ and approaches the WTP when $P$ is the only funder as $b_P \to 1$.

If $b_p < (\alpha_I - \alpha_P)/\alpha_I$, substitute $(1 - w)B_P$ for $B_P$ in the payoff expression from Proposition 9. Set the resulting expression equal to the payoff expression from Proposition 3 and solve for $w$.

By substituting $b_p = (\alpha_I - \alpha_P)/\alpha_I$ into the WTP expression, we see that, as one would expect, the WTP is continuous. If $\eta = 1$, we can straightforwardly see that it is decreasing in $b_p$ throughout the $b_p < (\alpha_I - \alpha_P)/\alpha_I$ interval by differentiating and finding that the resulting term is negative for all $b_p > 0$. We can also see that the WTP term tends to 1 as $b_p$ by taking the limit, applying L’Hôpital’s Rule. If $\eta \neq 1$, neither of these results is analytically tractable, but they are demonstrated numerically in the online appendix accompanying this paper.

**Proposition 12. Lower bound on WTP for patience given an altruistic impatient funder**

Given an altruistic impatient funder, if the funders engage in the defection equilibrium or a Pareto-superior equilibrium, a patient funder’s willingness to pay for patience is bounded below by the patient funder’s willingness to pay for patience when the patient funder is the only funder (the case of Proposition 3), which it approaches as $b_p \to 1$, and approaches 1 as $b_p \to 0$.

**Proof.** See Appendix A.7.

We saw in §3.2–3.3 that the presence of an impatient co-funder, warm-glow or altruistic, should often motivate a patient philanthropist to spend
even more slowly than he would if he were the only funder of a given public good. Here, perhaps unsurprisingly, we see that if the patient philanthropist spends at the impatient-optimal rate, his “error” (in willingness-to-pay terms) from spending impatiently is larger given the presence of the impatient co-funder. Furthermore, we see that as as the size of the impatient fund increases relative to his own, the error from spending impatiently grows arbitrarily large. That is, regardless of the other parameters, for sufficiently small $b_P$ he is willing to lose approximately his entire budget in exchange for the right to spend the remaining pittance patiently.

This finding underscores the potentially extreme undesirability of disbursement requirements from a patient perspective, discussed at the end of §2.3. It also highlights the value that relatively small patient donors may find in being more strategic with the timing of their giving. In practice, as noted at the end of §1, many large foundations seem eager to spend more patiently, but small donors generally seem to give away their charity funds as they are earned. However, small donors who are sympathetic to the moral case against pure time preference in the philanthropic context, and who hope to contribute to the funding of public goods primarily funded by larger and less patient actors, have the most to gain—indeed, in proportional terms, have arbitrarily much to gain—by putting their patience into practice.

4 Contribution matching

4.1 Motivation and framework

The continuum of efficient equilibria identified in Proposition 8 is a dynamic conditional contribution scheme, in which two (presumably at least relatively large) parties strategically condition their contributions on each other’s contribution histories.

In the face of smaller donors, it is common for large philanthropists not simply to contribute to public goods directly, but to encourage others to do so by explicitly matching their contributions in some proportion. When the large philanthropist is intent on contributing a fixed budget to his chosen cause, smaller impact-minded donors must worry that the matching is illusory; funds not used for contribution matching will ultimately be spent on the cause nonetheless (Kaufman, 2015). Here, however, we will posit that the large philanthropist has a lower rate of time preference $\delta_P$ than the small
donors’ $\delta_I$. Impatient donors therefore do have a counterfactual impact on the patient philanthropist’s behavior: funds not contributed at some time $t$ will still be spent on the same good, but not on the same schedule.

Throughout this section, we will assume that the impatient party (the “donor”) maximizes her utility taking the matching policy of the patient party (the “philanthropist”) as given. Equivalently, and perhaps more precisely, we will assume that the impatient party is not a single agent but a continuum of uncoordinated agents with identical preferences. That is, given an equilibrium in which a representative donor spends according to schedule $X_I(t)$ and her funds are matched accordingly, she can consider the value only of marginal shifts in resources across periods. She cannot consider shifts large enough to affect the affordability of the philanthropist’s matching policy. In this sense the analyses of §3 and §4 might be thought of as opposite ends of a spectrum, running from the case in which the patient philanthropist faces one impatient co-funder to the case in which he faces very many.

For realism and simplicity, we will restrict our attention to linear matching policies. Note that a linear matching policy at some ratio $M : 1$ is equivalent to a policy subsidizing the provision of the good by setting its price at $f = \frac{1}{1+M}$, where its price without the subsidy is normalized to 1. Also, subsidizing investment at $t$ is equivalent to proportionally subsidizing spending at all times $s > t$; so arbitrary linear contribution matching effectively encompasses the case in which the “philanthropist” subsidizes investment by the “donors”, as for example governments do by exempting foundations and DAFs from the capital gains tax.

### 4.2 Matching under discretion

In the philanthropic context, we must always have $f(t) \leq 1$. Before solving for the optimal price policy in this setting, however, let us consider the simpler case without this constraint. That is, let us consider the more familiar problem of a patient policymaker, who can both tax and subsidize others’ spending.

**Proposition 13. Optimal price schedule when taxation is feasible**

If the patient party can both tax and subsidize spending, he maximizes $\delta_P$-discounted utility by setting the effective price schedule

$$\hat{f}(t) = e^{(\delta_P-\delta_I)t} b_I \left(1 + \frac{\delta_I - \delta_P}{\alpha_P}\right).$$
Proof. See Appendix A.8.

Intuitively, a price schedule of the form $Ae^{-(\delta_I-\delta_P)t}$, for some $A$, lowers the beneficiary’s effective discount rate by $\delta_I - \delta_P$. Setting $A$ as high as is feasible thus produces the $\delta_P$-optimal spending plan for the collective budget. For each $t$, therefore, $\hat{f}(t)$ also produces the $\delta_P$-optimal spending plan for the budget remaining at $t$; that is, the price schedule is dynamically consistent. This result stands as a happy exception to the fact, well-known in the optimal taxation literature, that policymakers often cannot implement their first-best tax and subsidy policies over time without the power to commit (see e.g. Klein and Rios-Rull (2003); Benhabib and Rustichini (1997)).

Proposition 13 may offer useful advice for a patient policymaker in the setting described. Of course, however, a philanthropist cannot set $f(t) > 1$; a philanthropist cannot tax. If $b_P$ is large enough that $\hat{f}(0) \leq 1$, this constraint is never binding. (Observe that $\hat{f}(t)$ is decreasing in $t$.) Otherwise, however, the constraint does bind, so the problem changes as follows.

Given a price schedule $f$, let us define

$$X_f \triangleq \arg \max_{X_I} \int_0^\infty e^{-\delta_I t} u\left(\frac{X_I(t)}{f(t)}\right) dt : \int_0^\infty e^{-rt} X_I(t) dt \leq B_I. \quad (17)$$

That is, $X_f$ is the spending schedule that maximizes the donor’s utility given price schedule $f$.

If the philanthropist can commit to a subsidy plan at time 0, his problem is then to announce, at 0, a schedule $f : f(t) \leq 1$ of prices for the good such that

$$\int_0^\infty e^{-rt}(1 - f(t))X_f(t) dt \leq B_P, \quad (18)$$

and there is no $\bar{f}(t)$ satisfying this budget constraint such that $U_P(\bar{f}) > U_P(f)$, where

$$U_P(f) \triangleq \int_0^\infty e^{-\delta_P t} u\left(\frac{X_f(t)}{f(t)}\right) dt. \quad (19)$$

If the philanthropist cannot commit, however, we must explicitly posit not just a price schedule but a history-dependent price policy $F$, and this must be dynamically consistent. Informally, that is, we must require for every possible price history up to $t$ that, given that the philanthropist will
obey $F$ after $t$, it is not profitable for the philanthropist to deviate from $F$ at $t$. The philanthropist’s problem is then to announce, at 0, a price policy $F$ such that, given $X_{t,F}$, the budget and dynamic consistency constraints are both satisfied, and there is no $\tilde{F}$ satisfying both constraints such that $U(\tilde{F}) > U(F)$.

**Proposition 14. Discretionary price policy**

*TO FILL IN*

*Proof. See Appendix A.9.\qed*

### 4.3 Matching under commitment

**Proposition 15. Optimality of discretionary pricing when $\eta = 1$**

*TO FILL IN*

*Proof. See Appendix A.10.\qed*

**Proposition 16. Sub-optimality of discretionary pricing when $\eta \neq 1$**

*TO FILL IN*

*Proof. See Appendix A.11.\qed*

### 4.4 The value of patience

**Proposition 17. WTP for patient contribution matching**

*TO FILL IN*

*Proof. See Appendix A.12.\qed*

## 5 Applications

## 6 Conclusion

The incomplete model developed here suggests that many philanthropists could fulfill their own values substantially more effectively by spending more slowly. It is not clear whether further modeling would weaken or strengthen this suggestion, but we can at least conclude from our model so far that some
arguments given in favor of giving now are mistaken. In short, patient philanthropists should reconsider their decision to spend as quickly as they do—and philanthropically-concerned economists should consider that the problem of patience and philanthropy is important, decision-relevant, and amenable to further fruitful exploration.
References


Appendix

A.1 Proof of Proposition 1

Let

\[ Y(t) \triangleq e^{-rt}X(t) \]  \hspace{1cm} (20)

denote the resources allocated at time 0 for investment until, followed by spending at, \( t \). Let

\[ v_t(Y(t)) \triangleq e^{-\delta t}u(e^{rt}Y(t)) \]  \hspace{1cm} (21)

denote the discounted flow utility at \( t \) from allocation \( Y(t) \).

Since utility in spending is time-additive, differentiable, and strictly concave, allocation \( Y \) maximizes utility iff, for some constant \( k \),

\[
v' (Y(t)) = \frac{\partial}{\partial Y(t)} \left[ e^{-\delta t} \left( \frac{(e^{rt}Y(t))^{1-\eta} - 1}{1-\eta} \right) \right] = \lambda \ \forall t, \ \eta \neq 1;
\]

\[
= \frac{\partial}{\partial Y(t)} \left[ e^{-\delta t} \ln(e^{rt}Y(t)) \right] = \lambda \ \forall t, \ \eta = 1.
\]

Taking the derivative and rearranging, we have

\[ y(t) = \lambda^{\frac{1}{\eta}} e^{\frac{r - \delta - r\eta}{\eta} t}. \]

(22)

Subjecting this resource allocation to the budget constraint, we have

\[
\int_0^\infty \lambda^{\frac{1}{\eta}} e^{\frac{r - \delta - r\eta}{\eta} t} dt = B; \hspace{1cm} (23)
\]

\[ \lambda = \left( B \left( \frac{r\eta - r + \delta}{\eta} \right) \right)^{-\eta}. \]

(24)

Substituting (24) into (22), and recalling that \( X(t) = e^{rt}Y(t) \), we have

\[ X(t) = B \left( \frac{r\eta - r + \delta}{\eta} \right) e^{\frac{r - \delta}{\eta} t}. \]

(25)
A.2 Proof of Proposition 4

As in the proof of Proposition 1 (Appendix A.1), let

\[ Y_P(t) \triangleq e^{-rt}X_P(t) \]  

(26)

denote the resources the patient party allocates at time 0 for investment until, followed by spending at, \( t \). Let \( Y_I(t) \) be defined likewise, and let \( Y(t) \triangleq Y_P(t) + Y_I(t) \). Given that the impatient party follows allocation \( Y_I \), let

\[ v_{P,t}(Y(t)) \triangleq e^{-\delta_P t}u(e^{rt}Y(t)) \]  

(27)

denote the patient party’s discounted flow utility at \( t \) from allocation \( Y_P(t) \).

From Proposition 1, the impatient party’s spending schedule is

\[ Y_I(t) = B_I \alpha_I e^{(r - \alpha_I)t}, \]  

(28)

independently of \( Y_P \). Furthermore, the patient party’s discounted flow utility in the collective allocation \( Y(t) \) is time-additive, differentiable, strictly increasing, and strictly concave at each time \( t \). Taking \( Y_I(t) \) as given, therefore, the patient party maximizes his utility by setting \( Y_P(t) \) such that he is indifferent to marginal resource reallocation across times to which he is allocating resources at a positive rate, and weakly prefers marginal resource allocation to these times to marginal resource allocation to other times. That is, differentiating (27),

\[ \lambda_t(Y_P(t), Y_I(t)) \triangleq \frac{\partial}{\partial Y_P(t)} \left[ v_{P,t}(Y_P(t) + Y_I(t)) \right] \]  

(29)

\[ = e^{-\alpha_P \eta t}(Y_P(t) + Y_I(t))^{-\eta} \]

\[ = \lambda^* > 0 \quad \text{if } Y_P(t) > 0; \]

\[ \leq \lambda^* \quad \text{if } Y_P(t) = 0. \]

Substituting (28) into (29), we have that if \( Y_P(t) = 0 \),

\[ \lambda_t = (B_I \alpha_I)^{-\eta} e^{(\delta_I - \delta_P)t}. \]  

(30)

As we can see, if \( Y_P(t) = 0 \), \( \lambda_t \) is strictly increasing in \( t \). It follows from (29) that, if \( Y_P(t) > 0 \) for some \( t \), \( Y_P(s) > 0 \forall s > t \). That is, there is some \( t^* \) such that \( y_P(t) = 0 \forall t < t^* \) and \( Y_P(t) = 0 \forall t > t^* \).

Thus \( v_{P,t}(Y(t)) = \lambda^* \) is constant for all \( t > t^* \). This implies that following \( t^* \), the collective allocation \( Y(t)(t > t^*) \) constitutes the patient-optimal
allocation of the collective budget allocated to \( t > t^* \).

This leaves us with two cases.

If \( t^* = 0 \), then \( \lambda_t = \lambda^* \forall t \), so \( Y \) constitutes the patient-optimal allocation of the collective budget. The impatient allocation rate of \( B_I \) at \( t = 0 \) must therefore not be greater than the patient allocation rate of the collective budget at \( t = 0 \). That is,

\[
Y_I(0) = B_I \alpha_I \leq B \alpha_P.
\] (31)

If \( t^* > 0 \), note first that \( Y_P(t) \) must be continuous. If there were some \( \tilde{t} \) at which \( Y_P(t) \) were discontinuous, then, since \( Y_I(t) \) is continuous, \( Y(t) = Y_P(t) + Y_I(t) \) would also be discontinuous at \( \tilde{t} \). Because \( v_{P,t}(Y(t)) \) is continuous in \( t \) and \( Y(t) \), it too would then be discontinuous at \( \tilde{t} \). The patient party would then be able to increase his utility by reallocating marginal funds from \( \tilde{t} \) to \( \tilde{t} - \epsilon \) or \( \tilde{t} + \epsilon \), for some sufficiently small \( \epsilon > 0 \).

In particular, \( Y_P \) is continuous at \( t^* \). Since \( Y_P(t) = 0 \ \forall t < t^* \), it follows that \( Y_P(t^*) = 0 \).

Furthermore, since \( \lambda_t(Y_P(t), Y_I(t)) \) is continuous in \( Y_P(t), Y_I(t) \), and \( t \), and since \( \lambda_t(Y_P(t), Y_I(t)) = \lambda^* \ \forall t > t^* \), we now have \( \lambda_{t^*} = \lambda^* \). Thus \( Y(t^*) = Y_I(t^*) \) constitutes the patient-optimal allocation rate of the collective resources remaining at \( t^* \).

That is,

\[
B_I \alpha_I e^{-\alpha_I t^*} = (B_P + B_I e^{-\alpha_I t^*}) \alpha_P.
\] (32)

Rearranging, we have

\[
t^* = \ln \left( \frac{B_I \alpha_I - \alpha_P}{B_P \alpha_P} \right) / \alpha_I.
\] (33)

Now, multiplying both sides of (32) by \( e^{\alpha_I t^*} \) and substituting (33) for \( t^* \), we have

\[
B_I \alpha_I = \left( B_P \left( \frac{B_I \alpha_I - \alpha_P}{B_P \alpha_P} \right) + B_I \right) \alpha_P \>
B \alpha_P,
\] (34)

because, from (33),

\[
t^* > 0 \Rightarrow \frac{B_I \alpha_I - \alpha_P}{B_P \alpha_P} > 1.
\] (35)
The inequality on \( y_I(0) \) provided by (31) and (34) thus characterizes whether \( t^* = 0 \) or \( t^* > 0 \). In particular, solving it for \( t^* \), we have

\[
t^* = \begin{cases} 0 & \frac{B_I}{B_P} \frac{\alpha_I - \alpha_P}{\alpha_P} \leq 1 \\ \ln \left( \frac{B_I}{B_P} \frac{\alpha_I - \alpha_P}{\alpha_P} \right) \alpha_I & \frac{B_I}{B_P} \frac{\alpha_I - \alpha_P}{\alpha_P} > 1 \end{cases}
\]  

(36)

which reduces to

\[
t^* = \max \left( 0, \ln \left( \frac{B_I}{B_P} \frac{\alpha_I - \alpha_P}{\alpha_P} \right) \alpha_I \right).
\]  

(37)

Finally, observe that the collective budget at \( t^* \) is

\[
B_I e^{(r - \alpha_I) t^*} + B_P e^{r t^*}
\]  

(38)

in either case, and recall that \( X_P(t) \) will fill the gap between \( X_I(t) \) and the patient-optimal spending rate of the collective budget following \( t^* \). It follows immediately that

\[
X_P(t) = \begin{cases} 0, & t < t^* \\ \left( B_I e^{(r - \alpha_I) t^*} + B_P e^{r t^*} \right) \alpha_P e^{(r - \alpha_P)(t - t^*)} - B_I \alpha_I e^{(r - \alpha_I) t}, & t \geq t^* \end{cases}
\]
A.3 Proof of Proposition 5

Suppose
\[ \exists \bar{t}, \check{t} : X_P(t) > 0, X_I(\bar{t}) > 0. \]  
(39)

If
\[ (X_I(t) + X_P(t))^{-\eta} > e^{(r-I\alpha)}(X_I(\bar{t}) + X_P(\bar{t}))^{-\eta}, \]  
(40)

then I can do better to reallocate funding from \( \check{t} \) to \( \bar{t} \). Likewise, if
\[ (X_I(t) + X_P(t))^{-\eta} < e^{(r-P\alpha)}(X_I(\bar{t}) + X_P(\bar{t}))^{-\eta}, \]  
(41)

then P can do better to reallocate funding from \( \bar{t} \) to \( \check{t} \). Since \( \delta_I > \delta_P \), at least one of these two inequalities must obtain. Thus (39) cannot obtain in equilibrium.

It follows that there is some time \( t^* \) such that I is the sole funder for \( t < t^* \) and P is the sole funder for \( t > t^* \). Without substantive loss of generality (see Footnote 4), let us say that P funds the good at \( t^* \).

As in Proposition 1, observe that each party i does best to spend such that the \( \delta_i \)-discounted marginal value of spending at any time is equal to that of investing to spend at any subsequent time at which she spends. This will happen precisely when
\[ X_I(t) = L_I e^{(r-I\alpha)}t, \quad t \in [0, t^*); \]  
(42)
\[ X_P(t) = L_P e^{(r-P\alpha)}t, \quad t \in [t^*, \infty), \]
for some constants \( L_I, L_P \).

From the budget constraints
\[ \int_0^{t^*} L_I e^{-\alpha_I t} \, dt = B_I; \]  
(43)
\[ \int_{t^*}^{\infty} L_P e^{-\alpha_P t} \, dt = B_P, \]
we then have
\[ L_I = B_I \alpha_I \left( 1 - e^{-\alpha_I t^*} \right)^{-1}; \]  
(44)
\[ L_P = B_P \alpha_P e^{-\alpha_P t^*}. \]

Finally, observe that the spending rate must be continuous at \( t^* \); \( \lim_{t \uparrow t^*} X_I(t) = X_P(t^*) \). If the spending rate rose discontinuously at \( t^* \), P
would do better to reallocate some spending from \( t^* + \epsilon \) to \( t^* - \epsilon \) for some sufficiently small \( \epsilon > 0 \). Likewise, if the spending rate fell discontinuously, \( I \) would do better to reallocate marginal spending forward. We therefore have

\[
L_I e^{(r-I\alpha_I) t^*} = L_P e^{(r-P\alpha_P) t^*}
\]

\[
\implies t^* = \frac{1}{\alpha_I - \alpha_P} \ln \left( \frac{L_I}{L_P} \right).
\] (45)

Substituting (45) into (44) and simplifying, we get

\[
L_I = B_I \alpha_I + B_P \alpha_P; \quad (46)
\]

\[
L_P = B_P \alpha_P \left( 1 + \frac{B_I \alpha_I}{B_P \alpha_P} \right)^{\frac{\alpha_P}{\alpha_I}}.
\]

Finally, substituting (46) into (42) and (45), we have our final expressions for \( X_I(t) \) and \( X_P(t) \) and for \( t^* \) respectively.
A.4 Proof of Proposition 6

Let $X_{P,t}(X_I)$ denote the spending schedule that $P$ adopts, in equilibrium, given $I$’s choice of schedule $X_I(t)$. Let $T_P(X_I)$ denote $\{t : X_{P,t}(X_I) > 0\}$, and let $T_I(X_I)$ denote $\{t : X_I(t) > 0\}$. Of course, we cannot assume that these sets are disjoint. Let $Y(X_I) \leq B_I$ denote the budget that $I$ allocates to spending at times $t \in T_P(X_I)$.

As in Proposition 1, $P$ will spend such that the $\delta_P$-discounted marginal value of spending at any time $s \in T_P(X_I)$ is equal to that of investing to spend at any subsequent time $t \in T_P(X_I), t > s$. This will happen precisely when

$$X_P(t) = L_P e^{(r-\alpha_P)t} \forall t \in T_P(X_I),$$

(47)

for some constants $L_I, L_P$. Furthermore, $L_P$ will be chosen so that $X_P$ satisfies the budget constraint

$$\int_{T_P(X_I)} e^{-rt}(X_I(t) + X_{P,t}(X_I))dt = B_P + Y(X_I).$$

(48)

The resulting $\{X_I(t) + X_{P,t}(X_I)\}_{t \in T_P(X_I)}$ will constitute the unique $\delta_P$-optimal allocation of resources $B_P + Y(X_I)$ across $T_P(X_I)$.

Let $\tilde{T} \subset T_P(X_I)$ denote a set of times such that

$$\int_{\tilde{T}} e^{-rt}(X_I(t) + X_{P,t}(X_I))dt = Y(X_I).$$

(49)

Note that such $\tilde{T}$ must exist, by the continuity of total resource allocation with respect to time, and that if $Y(X_I) > 0$ any such $\tilde{T}$ must have positive measure.

Suppose $Y(X_I) > 0$, and consider an alternative schedule $\tilde{X}_I$ such that

$$\tilde{X}_I(t) = \begin{cases} 
X_I(t), & t \notin T_P(X_I); \\
X_I(t) + X_{P,t}(X_I), & t \in \tilde{T}; \\
0, & t \in T_P(X_I) \setminus \tilde{T}.
\end{cases}$$

(50)

$P$ will still be able to achieve a collective spending rate of $X_I(t) + X_{P,t}(X_I)$ across $T_P(X_I)$, by spending at rate

$$X_{P,t}(\tilde{X}_I) = \begin{cases} 
0, & t \notin T_P(X_I) \setminus \tilde{T}; \\
X_I(t) + X_{P,t}(X_I), & t \in T_P(X_I) \setminus \tilde{T}.
\end{cases}$$

(51)
This collective spending rate is still the unique $\delta_P$-optimal allocation of resources $B_P + Y(X_I)$ across $T_P(X_I)$. Furthermore, because $\bar{X}_I(t) = X_I(t) \forall t \notin T_P(X_I)$, $P$ still prefers spending within $T_P(X_I)$ to spending outside $T_P(X_I)$. Given $\bar{X}_I$, therefore, $P$ does indeed best respond with the schedule $X_P(\bar{X}_I)$ defined above.

$\bar{X}_I$ thus induces the same collective spending schedule as $X_I$, across all $t$. However, $Y(\bar{X}_I) = 0$. We have thus found that, for any spending schedule $X_I$, there is an alternative spending schedule $\bar{X}_I$ such that $Y(\bar{X}_I) = 0$ and such that $I$ is indifferent between $X_I$ and $\bar{X}_I$.

Consider a schedule $X_I$ such that $Y(X_I) = 0$, and suppose that there are positive-measure sets $T, \overline{T}$ such that $X_I(t) = 0 \forall t \in T$, $X_I(t) > 0 \forall t \in \overline{T}$, and $\sup(T) > \inf(\overline{T})$. Given any $\epsilon > 0$, we can choose $T, \overline{T}$ such that $T \subseteq (t - \epsilon, t)$, $\overline{T} \subseteq (\overline{t}, \overline{t} + \epsilon)$, and $\mu(\overline{T}) = e^{\alpha_P(\overline{t} - t)}\mu(T)$ (where $\mu$ denotes measure), for some times $t < \overline{t}$. Define a bijection $f : T \rightarrow \overline{T}$ such that $\mu(\text{Im} f(S)) = e^{\alpha_P(\overline{t} - t)}\mu(S)$ $\forall S \subseteq T$.

Consider an alternative schedule $\bar{X}_I$ such that

$$\bar{X}_I(t) = \begin{cases} X_I(t), & t \notin T \cup \overline{T}; \\ 0, & t \in T; \\ e^{-(r-\alpha_P)(f(t) - t)}X_I(f(t)), & t \in \overline{T}. \end{cases} \quad (52)$$

Let

$$Y_{P,T}(X_I) \equiv \int_T e^{-rt}X_{P,t}(X_I)dt \quad (53)$$

denote $P$’s allocation to some time or time-set $T$ given some $X_I$. ...

The next step of the proof is to show that $\sup(T_I(X_I)) \leq \inf(T_P(X_I))$. I have a tedious proof sketch of this on paper, but this proof is much easier if we can approximate continuous time with a certain nonstandard grid sequence. I am looking into whether this permissible.

The rest of the proof then proceeds like the first half of the proof of Proposition 7 (Appendix A.5). I intend to move that up to this section.
A.5 Proof of Proposition 7

Given a strategy profile $\sigma$, let $t^* \triangleq \min(\{t : B_I(X(\sigma)_t) = 0\})$. In a defection equilibrium $\sigma^D$ (if one exists), $t^* < \infty$, and $P$ spends nothing until $t^*$. It is then clear that, starting at $t^*$, $P$ will allocate his resources patient-optimally, regardless of what $I$ has done up to $t^*$. So

$$
\sigma^D_P(X_{|t}) = \begin{cases} 
0 & B_I(X_{|t}) > 0; \\
B_P(X_{|t})\alpha_P & B_I(X_{|t}) = 0.
\end{cases}
$$ (54)

Given this $\sigma^D_P$, a strategy $\sigma_I$ is a best response iff, at every node $X_{|s}$ at which $B_I(X_{|s}) > 0$, we obtain the $t^* > s$ and \{X_{I,t}\}_{t \in [s, t^*)}$ that maximizes

$$
\int_s^{t^*} e^{-\delta_I(t-s)} u(X_{I,t}) dt + \int_{t^*}^{\infty} e^{-\delta_I(t-s)} u(X_{P,t}) dt,
$$ (55)

subject to

$$
\int_s^{t^*} e^{-r(t-s)} X_{I,t} dt \leq B_I(X_{|s}),
$$ (56)
given that

$$
X_{P,t} = \begin{cases} 
0 & t \in [s, t^*);
\\
B_P(X_{|s})e^{r(t^*-s)}\alpha_P e^{(r-\alpha_I)(t-t^*)}, & t \geq t^*.
\end{cases}
$$ (57)

We will assume, without substantive loss of generality, that $X_{I,t^*} = 0$.

As in the proof of Proposition 1 (Appendix A.1), observe that $I$ does best to spend such that the marginal value of spending at any time $s$ is equal to that of investing to spend at any subsequent time $t$ during which she spends, and that this will happen precisely when

$$
X_{I,t} = Le^{\frac{-\delta_I}{\sigma}t},
$$ (58)

for some constant $L$, across the times $t \in [s, t^*)$ during which she will spend. Subjecting this schedule to the budget constraint, given some $t^*$, we have

$$
\int_s^{t^*} Le^{-r(t-s)}e^{(r-\alpha_I)t} dt = B_I(X_{|s})
$$ (59)

$$
\Rightarrow L = B_I(X_{|s})\alpha_I e^{(r-\alpha_I)s} \left(1 - e^{\alpha_I(t^*-s)}\right)^{-1}
$$

$$
\Rightarrow X_{I,t} = B_I(X_{|s})\alpha_I \left(1 - e^{\alpha_I(t^*-s)}\right)^{-1} e^{r(t^*)} e^{(r-\alpha_I)(t-s)}, \quad t \in [s, t^*].
$$
From (55), (59), and (57), the utility $I$ attains from spending her budget optimally by $t^*$ is therefore

$$\int_s^{t^*} e^{-\delta_I(t-s)} \left( \left( B_I(X_{|s}) \alpha_I \left( 1 - e^{\alpha_I(t^*-s)} \right)^{-1} e^{(r-\alpha_I)(t-s)} \right)^{1-\eta} - 1 \right) dt$$

$$+ \int_{t^*}^{\infty} e^{-\delta_I(t-s)} \left( B_P(X_{|s}) e^{\gamma(t^*-s)} \alpha_P e^{(r-\alpha_I)(t-t^*)} \right)^{1-\eta} - 1 \right) dt \cdot \frac{1}{1-\eta}, \quad \eta \neq 1;$$

$$\int_s^{t^*} e^{-\delta_I(t-s)} \ln \left( B_I(X_{|s}) \delta_I \left( 1 - e^{-\delta_I(t^*-s)} \right)^{-1} e^{(r-\delta_I)(t-s)} \right) dt$$

$$+ \int_{t^*}^{\infty} e^{-\delta_I(t-s)} \ln \left( B_P(X_{|s}) e^{\gamma(t^*-s)} \delta_P e^{(r-\delta_P)(t-t^*)} \right) dt, \quad \eta = 1. $$

Simplifying and integrating these terms gives

$$\frac{B_I(X_{|s})^{1-\eta}}{1-\eta} \alpha_I^{-\eta} \left( 1 - e^{-\alpha_I(t^*-s)} \right)^{\eta}$$

$$+ \frac{B_P(X_{|s})^{1-\eta}}{1-\eta} \left( \alpha_P \right)^{1-\eta} \frac{1}{\gamma \alpha_I} e^{-\alpha_I(t^*-s)} - \frac{1}{\delta_I(1-\eta)}, \quad \eta \neq 1;$$

$$\frac{e^{-\delta_I(t^*-s)}}{\delta_I} \left( \ln \left( B_P(X_{|s}) \delta_P \right) - \frac{\delta_P}{\delta_I} + \delta_I(t^*-s) + 1 \right)$$

$$+ \frac{e^{-\delta_I(t^*-s)}}{\delta_I} - 1 \ln \left( \frac{1 - e^{-\delta_I(t^*-s)}}{B_I(X_{|s}) \delta_I} \right) - \frac{1}{\delta_I} + \frac{r}{\delta_I^3}, \quad \eta = 1.$$

By these terms’ first-order conditions with respect to $t^*$, we find a unique maximum at

$$t^* = \ln \left( Z(B_I(X_{|s}), B_P(X_{|s}))/\alpha_I + s, \right)$$

where

$$Z(B_I(X_{|s}), B_P(X_{|s})) \triangleq \begin{cases} 
1 + \frac{B_I(X_{|s}) \alpha_I}{B_P(X_{|s}) \alpha_P} \gamma, & \eta \neq 1; \\
1 + \frac{B_I(X_{|s}) \delta_I}{B_P(X_{|s}) \delta_P} e^{\frac{s_P}{\eta} - 1}, & \eta = 1.
\end{cases}$$

By construction, $\sigma^D_I$ is $I$’s unique best response to $\sigma^D_P$. We will now show that $\sigma^D_P$ is $P$’s unique best response to $\sigma^D_I$. 
Given $\sigma^P$, suppose $P$ spends $X_{P,s} \geq 0$ at some time $s < t^*$, and follows strategy $\sigma^P$ at all $t \neq s$. The utility $\tilde{P}$ attains across $t \geq s$ is

$$
\begin{align*}
&(X_{I,s} + X_{P,s})^{1-\eta} - 1 \ dt \\
&+ \int_s^{t^*} e^{-\delta P(t-s)} \left( \left( (B_I(X_{I,s}) - X_{I,s}dt)e^{\gamma dt} \alpha_I \left( 1 - e^{-\alpha_I(t-(s+dt))} \right)^{1-\eta} - 1 \right) dt \right) + \int_s^{\infty} e^{-\delta P(t-s)} \left( (B_P(X_{P,s}) - X_{P,s}dt)e^{\gamma dt} \alpha_P e^{\frac{\gamma dt}{r}(t-t')} \right)^{1-\eta} - 1 \ dt \right) \frac{1}{1-\eta}, \eta \neq 1; \\
&\ln((X_{I,s} + X_{P,s})dt) \\
&+ \int_s^{t^*} e^{-\delta P(t-s)} \ln \left( (B_I(X_{I,s}) - X_{I,s}dt)e^{\gamma dt} \delta_I \left( 1 - e^{-\delta_I(t-(s+dt))} \right)^{1-\eta} - 1 \right) dt \\
&+ \int_s^{\infty} e^{-\delta P(t-s)} \ln \left( (B_P(X_{P,s}) - X_{P,s}dt)e^{\gamma dt} \delta_P e^{(t-t')} \right)^{1-\eta} - 1 \ dt, \quad \eta = 1.
\end{align*}
$$

As our expression for $X_{I,s}$, we will use (59), with $t = s$ and

$$
t^* = \ln(z) / \alpha_I + s, \quad z = Z(B_I(X_{I,s}), B_P(X_{I,s})).
$$

In the integrals, we will use

$$
t^* = \ln(\tilde{z}) / \alpha_I + s + dt, \\
\tilde{z} = Z((B_I(X_{I,s}) - X_{I,s}dt), (B_P(X_{I,s}) - X_{P,s}dt).
$$

Then we will simplify and integrate, getting

$$
\begin{align*}
&(B_I(X_{I,s}) \alpha_I \tilde{z}^{\frac{1}{1-\eta}} + X_{P,s})^{1-\eta} \\
&+ \left[ (1 - \tilde{z}^{\frac{\delta_I - \delta_P}{\alpha_I}} - 1) (B_I(X_{I,s}) \left( 1 - \frac{\alpha_I}{z - 1} dt \right) \alpha_I \frac{\tilde{z}}{z - 1})^{1-\eta} \frac{1}{\alpha_I - \delta_I + \delta_P} \\
&+ \tilde{z}^{-\alpha_I \eta} (B_P(X_{I,s}) - X_{P,s}dt)^{1-\eta} \alpha_P^{-\eta} \frac{e^{-\alpha_P \eta dt}}{1-\eta} - \frac{1}{\delta_P(1-\eta)} \right], \eta \neq 1; \\
&\ln((B_I(X_{I,s}) \delta_I \tilde{z}^{\frac{1}{1-\eta}} + X_{P,s})dt) \\
&+ \left[ (1 - \tilde{z}^{\frac{\delta_P}{\alpha_P}}) \ln((B_I(X_{I,s}) \left( 1 - \frac{\delta_I}{z - 1} dt \right) \delta_I \frac{\tilde{z}}{z - 1} - \frac{\delta_I}{\delta_P}) \\
&+ \tilde{z}^{\frac{\delta_P}{\alpha_P}} \ln((B_P(X_{I,s}) - X_{P,s}dt)\delta_P) + \frac{r}{\delta_I} \ln(\tilde{z}) - 1 + \frac{r}{\delta_P} + r dt \right] e^{-\delta_P dt} \delta_P, \eta = 1.
\end{align*}
$$
Now we will differentiate with respect to $X_{P,s}$ (recalling that $X_{P,s}$ appears in $\tilde{z}$); divide the resulting term by $dt$, to rescale the payoffs from deviating from infinitesimal absolute payoffs to nonzero payoffs per unit time; and set $dt$ to 0. We then have the rescaled payoff to an instantaneous deviation:

$$
\left(\frac{B_P\alpha_P}{\alpha}\right)^{-\eta}\left(\frac{\alpha_P}{\gamma(\alpha_I - \delta_I + \delta_P)} - \frac{B_I}{B_P(\eta - 1)}\right)z^{-\frac{B_P\eta}{\alpha-I}} - 1\right)z^{-\frac{B_P\eta}{\alpha-I}}
+(B_P\alpha_P)^{-\eta}\left(\frac{B_I\alpha_P\eta}{(\eta - 1)(B_I\alpha_I\eta + B_P\alpha_P)} - 1\right)z^{-\frac{B_P\eta}{\alpha-I}}
- \frac{B_P\eta}{\alpha_I - \delta_I + \delta_P}\left(\frac{\alpha_P}{\gamma}\right)^{1-\eta}z^{-\eta} + \left(\alpha_I B_I + \frac{B_P\alpha_P}{\gamma} + X_{P,s}\right)^{-\eta}, \eta \neq 1;
$$

$$
\left[\frac{e^\delta_P}{\delta_P-\gamma}\left(\frac{B_I (X_{I,s})}{B_P(X_{P,s})}\right)^{\alpha_P}\gamma^{n+1}\left(\frac{\alpha_I - \delta_I + \delta_P}{\alpha_P}\right)\right]z^{-\frac{B_P\eta}{\alpha-I}} - 1 - \frac{B_P\eta}{\alpha_I - \delta_I + \delta_P}\gamma^{n} + 1.
$$

As we can see, the total payoff impact can be split into the additional flow utility from spending at $s$—the last term in both expressions above—and the utility impact from adjusting $I$'s spending schedule, and thus $t^*$ and ultimately $P$'s spending schedule, after $s$. The former is decreasing in $X_{P,s}$, and the latter is independent of $X_{P,s}$. To verify that $P$ never wants to deviate with any positive spending rate at $s$, therefore, we only have to verify that the expressions above are always negative at $X_{P,s} = 0$. In the $\eta \neq 1$ case, this is equivalent to the condition that

$$
\frac{1}{\gamma^{n+1}\left(\frac{\alpha_I - \delta_I + \delta_P}{\alpha_P}\right)}\left(1 + \frac{B_I (X_{I,s})}{B_P(X_{P,s})}\right)^{\alpha_P}\gamma^{n+1}\left(\frac{\alpha_I - \delta_I + \delta_P}{\alpha_P}\right)\right]z^{-\frac{B_P\eta}{\alpha-I}} - 1 - \frac{B_P\eta}{\alpha_I - \delta_I + \delta_P}\gamma^{n} + 1.
$$

In the $\eta = 1$ case, this is equivalent to the condition that

$$
\frac{B_I (\delta_I + \delta_P)}{B_P} z^{-\delta_P}\geq 1 - e^{1-\delta_P}.
$$

If $\eta < 1$,

$$
\frac{\alpha_I - \alpha_P - \delta_I + \delta_P}{\alpha_I - \delta_I + \delta_P} > 0
$$

and

$$
\frac{n + 1}{n - 1} \frac{\alpha_I - \delta_I + \delta_P}{\alpha_I - \alpha_P - \delta_I + \delta_P} > 1.
$$
Recall our formula for $z$ from (69) and (65), and observe that $z$, and thus $z$ to any power, is positive. It follows that the left hand side of (76) is greater than

$$\frac{1}{\gamma} \frac{\alpha_I - \alpha_P - \delta_I + \delta_P}{\alpha_I - \delta_I + \delta_P} z^{-\frac{\alpha_P}{\alpha_I}} \eta.$$  \hspace{1cm} (78)

Observe that $z > 1$ and that, when $\eta < 1$, $\eta - \frac{\alpha_P}{\alpha_I} \eta > 0$. Since $\gamma < 1$, from here it is easy to verify that condition (76) holds.

If $\eta > 1$, twice differentiate the left hand side of (76) with respect to $\frac{B_I}{B_P}$, recalling that this term appears in $z$ but nowhere else in the expression. (I am supressing the “$X_i$s” argument here for clarity.) We find that the expression has a unique global minimum at

$$\frac{B_I}{B_P} = \frac{\alpha_P}{\alpha_P - \alpha_I} \gamma \left( \frac{1}{\eta} + \frac{\eta - 1}{\eta + 1} (1 - \alpha_I + \delta_I - \delta_P) \right).$$  \hspace{1cm} (79)

If (79) is nonpositive, then as long as (76) holds at $\frac{B_I}{B_P} = 0$, it will hold at all $\frac{B_I}{B_P} > 0$. So we must simply evaluate (76) after substituting 0 for $\frac{B_I}{B_P}$. If (79) is positive, we must evaluate (76) after substituting (79) for $\frac{B_I}{B_P}$. In both cases, we find that the inequality holds.

If $\eta = 1$, it follows from $\delta_I > \delta_P$ that the left-hand side of (77) is positive and that the right-hand side is negative. Thus the inequality holds.

By the one-shot deviation principle in continuous time, $\sigma^D_P$ is a best response to $\sigma^D_I$.

We have now shown that $\sigma^D$ is an equilibrium. Furthermore, the definition of defection equilibrium determines $P$’s strategy, and we have found $I$’s unique best response to this strategy, so $\sigma^D$ must be the unique defection equilibrium.

Finally, substituting (64) into (59) at $s = 0$ gives our spending schedule result for $X_{I,t}$, and substituting (64) into (57) at $s = 0$ gives our spending schedule result for $X_{P,t}$. 

A.6 Proof of Proposition 8

An efficient spending schedule $X$ maximizes

$$U_a(X) \triangleq aU_I(X) + (1-a)U_P(X)$$  \hspace{1cm} (80)

for some weight $a \in [0,1]$. Given efficient spending schedule $X$, therefore, the corresponding $U_a$ cannot be increased by moving resources between time 0 and any other time $t$. That is,

$$U'_a(X_0) = e^{rt}U'_a(X_t)$$

$$\implies X_0^{-\eta} = e^{rt}(ae^{-\delta I t} + (1-a)e^{-\delta P t})X_t^{-\eta}.$$  \hspace{1cm} (81)

That is, $X$ is optimal according to time preference factor

$$\beta_a(t) = ae^{-\delta I t} + (1-a)e^{-\delta P t},$$  \hspace{1cm} (82)

or time preference rate

$$\delta_a(t) = \frac{-\beta'(t)}{\beta(t)} = \frac{a\delta_I e^{-\delta I t} + (1-a)\delta_P e^{-\delta P t}}{ae^{-\delta I t} + (1-a)e^{-\delta P t}}.$$  \hspace{1cm} (83)

As we can see, $\delta_a(0) = a\delta_I + (1-a)\delta_P$. Therefore $a$ is not only the weight placed on $I$’s utility, but also the weight placed on her time preference in determining the starting time preference rate.

Let $w_a(t)$ denote the weight placed on $I$’s time preference rate at time $t$, such that

$$\delta(t) = w_a(t)\delta_I + (1-w_a(t))\delta_P.$$  \hspace{1cm} (84)

(As we can see, $w_a(0) = a$.) Substituting (83) into (84) and rearranging, we have

$$w_a(t) = \frac{a}{a + (1-a)e^{(\delta_I-\delta_P)t}}.$$  \hspace{1cm} (85)

Having fixed weight $a$ to place on $I$’s forward-looking utility, the resulting spending schedule is not time-consistent, because the resulting time preference rate is not constant. Upon reaching each $s > 0$, $aU_I(X) + (1-a)U_P(X)$ can be maximized across times $t > s$ by following time preference rate schedule $\delta(t-s)$ rather than $\delta(t)$ as prescribed.
However, if upon reaching \( s \) we instead place weight
\[
\tilde{a} = w_a(s)
\] (86)
on \( I \)'s forward-looking utility, the resulting time preference rate schedule is the same for \( t > s \) as that prescribed at time 0 using weight \( a \). That is,
\[
\delta_{\tilde{a}}(t - s) = \delta_a(t) \quad \forall t \geq s.
\] (87)
We can see this by substituting (86) into (85) and the result into (82), simplifying, and differentiating:
\[
\beta_{\tilde{a}}(t - s) = \frac{a \cdot e^{-\delta_I(t-s)}}{a + (1-a)e^{(\delta_I-\delta_P)s}} + \frac{(1-a)e^{(\delta_I-\delta_P)s}}{a + (1-a)e^{(\delta_I-\delta_P)s}} e^{\delta_Is}.
\] (88)
\[
\Longleftrightarrow \delta_{\tilde{a}}(t - s) = \frac{\beta'(t - s) \cdot a \delta_I e^{-\delta_I t} + (1-a)\delta_P e^{-\delta_P t}}{e^{-\delta_I t} + (1-a)e^{-\delta_P t}} = \delta_a(t).
\]
Let \( X(a) \) be the efficient spending schedule implied by weight \( a \), and let \( x(a) \) be its normalization. Given \( b_I \in (0,1) \), \( X \) is a Pareto improvement to the defection schedule \( X^{D} \) iff \( x(a) \) is a Pareto improvement to the normalized defection schedule \( x^{D}(b_I) \):
\[
U_i(X(a)) \geq U_i(X^{D})
\] (89)
\[
\Longleftrightarrow B^{1-\eta}U_i(x(a)) + \frac{B^{1-\eta} - 1}{\delta_I(1-\eta)} \geq B^{1-\eta}U_i(x^{D}) + \frac{B^{1-\eta} - 1}{\delta_I(1-\eta)} \quad \eta \neq 1;
\]
\[
U_i(x(a)) + \ln(B) \geq U_i(x^{D}) + \ln(B) \quad \eta = 1;
\]
\[
\Longleftrightarrow U_i(x(a)) \geq U_i(x^{D}),
\]
with the same of course holding for strict inequalities.

Given any efficient normalized schedule \( x(a) \), for some weight \( a \in (0,1) \), there is some range of values \( [b_I, \bar{b}_I] \subset (0,1) \) such that \( x(a) \) is a Pareto improvement on \( x^{D}(b_I) \) iff \( b_I \in [b_I, \bar{b}_I] \). This follows directly from the inefficiency of \( x^{D}(b_I) \) for \( b_I \in (0,1) \) and the facts that
- \( U_I(x^{D}(0)) = U_I(x(0)) \),
PATIENCE AND PHILANTHROPY

\[ U_P(x^D(0)) = U_P(x(0)), \]
\[ U_I(x^D(1)) = U_P(x(1)), \]
\[ U_P(x^D(1)) = U_P(x(1)), \]
\[ U_I(x^D(b_I)) \text{ is continuous and monotonically increasing in } b_I, \text{ and} \]
\[ U_P(x^D(b_I)) \text{ is continuous and monotonically decreasing in } b_I. \]

We can thus define

\[ b_I(a) \triangleq \arg \min_{b_I} : U_P(x(a)) \geq U_P(x^D(b_I)), \]
\[ \overline{b_I}(a) \triangleq \arg \max_{b_I} : U_I(x(a)) \geq U_I(x^D(b_I)). \]

By construction, \([b_I(a), \overline{b_I}(a)]\) is the range of budget proportions \(b_I\) initially belonging to \(I\) such that \(x(a)\) is a Pareto improvement on \(x^D\). As shown above, it is also the \(b_I\)-range such that \(X(a)\) is a Pareto improvement on \(X^D\). So both parties to weakly prefer cooperation to defection at \(t = 0\) iff \(b_I \in [b_I(a), \overline{b_I}(a)]\).

More generally, given a strategy profile \(\sigma\), both parties weakly prefer the forward-looking spending schedule \(\{X_s(a)\}_{s \geq t}\) to defection at all times \(t\) iff

\[ b_I(x|t(\sigma)) \in [b_I(w_a(t)), \overline{b_I}(w_a(t))] \forall t \geq 0. \]

(91)

Given \(b_I\), consider a strategy profile \(\sigma^*\) that implements a Pareto improvement \(X(a)\) to \(Bx^D(b_I)\), and suppose that

\[ \sigma^*_i(X|t) = \sigma^*_D(X|t) \forall X|t \neq X|t(\sigma^*) \forall i. \]

(92)

That is, if either party defects from \(\sigma^*\), they both subsequently follow the defection equilibrium. Since \(b_I \in [b_I(a), \overline{b_I}(a)]\), relation (91) holds for \(t = 0\).

If \(\sigma^*\) maintains condition (91) at all \(t\), then \(\sigma^*\) is an equilibrium.

Since \(b_I(x|t(\sigma))\), \(b_I(x|t(\sigma))\), and \(\overline{b_I}(x|t(\sigma))\) are all continuous in \(t\) for any utility weight \(a\) and any strategy profile \(\sigma\), we only need to show that \(\sigma^*\) can be constructed such that \(b_I(x|t(\sigma))\) never crosses \(b_I(x|t(\sigma))\) or \(\overline{b_I}(x|t(\sigma))\). We will now show this, by contradiction.

Consider a time \(t\) such that \(b_I(x|t(\sigma^*)) = \overline{b_I}(x|t(\sigma^*))\). That is, suppose that, by following \(\sigma^*\), a time \(t\) comes when \(I\) is indifferent between defecting
and continuing to follow $\sigma^*$. Now, specify that $\sigma^*_I(x|_{t\varepsilon}(\sigma^*)) = X_\varepsilon(a)$ for $s \in [t, t+\varepsilon)$, for some $\varepsilon > 0$ small enough that $b_I(x|_{t+\varepsilon}(\sigma^*)) \leq b_I(w_a(t+\varepsilon))$. That is, define $\sigma^*$ such that, for length $\varepsilon$ of time following $t$, $I$ contributes the entirety of the spending. Then, for all $s \in (t, t+\varepsilon)$, $I$ strictly prefers to maintain $\sigma^*$ than to defect. To see this, observe that if the forward-looking defection payoff for $I$ at $t+\varepsilon$ were higher than the forward-looking payoff from $\sigma^*$ at $t+\varepsilon$, then, at $t$, the payoff to spending at rate $X_\varepsilon(\sigma^*)$ for $s \in [t, t+\varepsilon)$ followed by defection at $t + \varepsilon$ would exceed the payoff from $\sigma^*$. But, by construction, the defection payoff for $I$ at $t$ is the best $I$ can get at $t$ given that $P$ will not spend until $b_I = 0$, which holds in either case. It would then follow that, at $t$, $I$ prefers defection to following $\sigma^*$, in contradiction to our assumption.

Likewise, consider a time $t$ such that $b_I(x|_t(\sigma^*)) = b_I(x|_{t+\varepsilon}(\sigma^*))$. That is, suppose that, by following $\sigma^*$, a time $t$ comes when $P$ is indifferent between defecting and continuing to follow $\sigma^*$. Now, specify that $\sigma^*_P(x|_s(\sigma^*)) = X_\varepsilon(a)$ for $s \in [t, t+\varepsilon)$, for some $\varepsilon > 0$ small enough that $b_I(x|_{t+\varepsilon}(\sigma^*)) \geq b_I(w_a(t+\varepsilon))$. That is, define $\sigma^*$ such that, for length $\varepsilon$ of time following $t$, $P$ contributes the entirety of the spending. Then $b_I(x|_s(\sigma^*))$ rises from $s = t$ to $t + \varepsilon$. Meanwhile, because $w_a(s)$ falls with time (see (85)), $U_P(w_a(s))$ rises with time; a normalized unit of resources is allocated efficiently in a way that places ever less weight on $I$’s forward-looking utility and ever more weight on $P$’s. And $U_P(x_D(b_I))$ decreases in $b_I$ (as is intuitive, and can be seen formally by differentiating the expression from Proposition 10 with respect to $b_I$ at $B = 1$). It thus follows from the definition of $b_I(a)$ (see (90)) that $b_I(w_a(s))$ falls over time. With $b_I(x|_s(\sigma^*))$ rising and $b_I(w_a(s))$ falling, it follows that condition (91) is maintained up to $t + \varepsilon$.

In short, if $b_I$ gets close to the upper end of the range, $\sigma^*$ can require $I$ to contribute a larger share of flow spending, and if $b_I$ gets close to the lower end, $\sigma^*$ can require $P$ to contribute a larger share. Having constructed $\sigma^*$ such that there is no time $t$ at which $b_I(x|_t(\sigma))$ crosses the necessary thresholds, (91) is always maintained, and $\sigma^*$ is an equilibrium.
A.7 Proof of Proposition 12
A.8 Proof of Proposition 13

Facing price schedule \( f(t) \), let us denote the donor’s budget allocation \( Y_f(t) \). The allocation is then impatient-optimal iff for some constant \( \lambda \),

\[
\frac{\partial}{\partial Y_f(t)} \left[ e^{-\delta_I u} \left( \frac{e^{rt} Y_f(t)}{f(t)} \right) \right] = \lambda \forall t. \tag{93}
\]

Taking the derivative, substituting \( \hat{f} \) for \( f \), and rearranging, we have

\[
Y_{\hat{f}}(t) = \lambda \frac{1}{\eta} e^{(\delta_P - \delta_I - \alpha_P) t} b_I^{\frac{\eta - 1}{\eta}} \left( 1 + \frac{\delta_I - \delta_P}{\alpha_P} \right)^{\frac{\eta - 1}{\eta}}. \tag{94}
\]

Subjecting this resource allocation to the donor’s budget constraint, we have

\[
\lambda \frac{1}{\eta} b_I^{\frac{\eta - 1}{\eta}} \left( 1 + \frac{\delta_I - \delta_P}{\alpha_P} \right)^{\frac{\eta - 1}{\eta}} \int_0^\infty e^{(\delta_P - \delta_I - \alpha_P) t} dt = B_I; \tag{95}
\]

\[
\lambda \frac{1}{\eta} b_I^{\frac{\eta - 1}{\eta}} \left( 1 + \frac{\delta_I - \delta_P}{\alpha_P} \right)^{\frac{\eta - 1}{\eta}} = B_I (\delta_I - \delta_P + \alpha_P). \tag{96}
\]

Substituting (96) into (94) and remembering that \( X(t) = e^{rt} Y(t) \), we see that the donor spends according to schedule

\[
X_{\hat{f}}(t) = B_I (\delta_I - \delta_P + \alpha_P) e^{(r + \delta_P - \delta_I - \alpha_P) t}. \tag{97}
\]

At each \( t \), collective spending thus equals

\[
\frac{X_f(t)}{\hat{f}(t)} = B_{\alpha_P} e^{(r - \alpha_P) t}. \tag{98}
\]

From Proposition 1, this is the spending schedule which maximizes \( \delta_P \)-discounted utility given budget constraint \( B \). It follows both that the given price schedule \( \hat{f} \) is feasible and there is no patient-preferable feasible price schedule.
A.9 Proof of Proposition 14
A.10 Proof of Proposition 15
A.11 Proof of Proposition 16
A.12 Proof of Proposition 17