Dynamic Public Good Provision
under Time Preference Heterogeneity:
Theory and Applications to Philanthropy

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Abstract

I explore the implications of time preference heterogeneity for public good funding. I find that the assumption of a common discount rate is knife-edge: allowing for time preference heterogeneity produces substantially different funding behavior in equilibrium. In particular I find that, across a variety of circumstances, patient funders invest, rather than spend, the entirety of their resources for substantial lengths of time in equilibrium. I also find that the implications of this departure from the common-discount-rate case are economically significant, in that the patient payoff to spending in equilibrium, relative to that of spending according to an intermediate time preference rate, can grow arbitrarily large as a patient funder's share of initial funding goes to zero. Finally, I discuss applications of these results to the timing of philanthropic spending, and to patient philanthropists' willingness to pay to avoid legal disbursement minima.

1 Introduction

The modern literature on public good contribution games begins with Bergstrom et al. (1986). Bergstrom et al. observed that well-behaved static public good contribution games feature a unique equilibrium allocation of resources to particular public goods, in which each individual is indifferent to marginal reallocations of resources among the goods she herself funds and weakly prefers reallocations to goods she funds from goods she does not. Even individuals with relatively similar preferences thus typically find themselves "polarized", in the sense that they fund entirely or almost entirely non-overlapping sets of projects. Similar observations have been made, seemingly independently, on other occasions: see e.g. Kalai and Kalai (2001).

There is now an extensive literature on dynamic public good contribution games as well. A central concern of this literature is efficiency: that is, roughly, determining when the threats made possible by repeated interaction can induce players to avoid free-riding on each other's contributions, substituting public good spending for spending on private consumption. It is well understood that in a simple continuous-time model under perfect monitoring, fully efficient spending schedules can obtain in subgame-perfect equilibrium, since the gains from deviation are infinitesimal relative to the losses from future punishments. As a result, authors have explored a vast array of modifications to the simple model, including imperfect monitoring, shocks, and investment irreversibility.

Despite its size, however, the literature on dynamic public good contribution games near-universally assumes that the actors under consideration are equally patient: i.e. that they act under a common discount rate. This common-discounting assumption is pervasive both in the theoretical literature and in applied work on the dynamic provision of public goods, such as country-level efforts to mitigate climate change; see e.g. Ferrari et al. (2015) or Dutta (2017). Fishman (2019) explores bargaining over public good provision in a dynamic setting where the players use different discount rates, and a small literature building on Sorger (2006) likewise explores the implications of bargaining under time preference heterogeneity in dynamic settings, some of which could apply to public good provision problems. Finally, the only other paper on the theory of public good contribution games even to mention time preference heterogeneity, of which I am aware—and so the only paper to do so outside a bargaining context—is Jacobsen et al. (2017), but it is set in a static environment: lower time preference, i.e. greater concern for the future, is simply listed as one reason why individuals may have different preferences regarding the provision of an environmental good in the present.

A deeper analysis of dynamic public good provision under time preference heterogeneity is valuable, I will argue, for at least two reasons.

First, individual rates of time preference vary widely.[1] Most developed-world gov-

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[1] A helpful review of the econometric and experimental literature on time preference heterogene-

ernments publish discounting guidelines that make explicit the discount rates they use in cost-benefit analysis for public policy, revealing unambiguously that they too act under heterogeneous rates of time preference.[2] Economists' recommendations of time preferences to use in social discounting differ substantially.[3] Philanthropists' time preferences appear to vary as well, both with each other and with those of individuals and policymakers. Of course, individuals, policymakers, and philanthropists all regularly contribute to public goods to which other such parties also contribute, and these parties must all decide how to allocate their contributions over time. In doing so, they participate in dynamic public good contribution games. Real-world dynamic public good contribution games likely, therefore, exhibit substantial time preference heterogeneity. Our attempts to model these games, and improve public good provision processes in light of them, will likely fail if we do not account for it.

Second, in practice, many individuals currently hold philanthropically-purposed assets in tax-exempt vehicles where they are earmarked for future charitable giving, such as DAFs. Assets in DAFs in particular, in the United States, currently total almost $150 billion; contributions to them have historically grown at a substantially higher rate than charitable contributions as a whole; and disbursements have not risen as quickly as contributions (National Philanthropic Trust, 2020). As we will see, this pattern can straightforwardly be explained as rational behavior by patient philanthropists given "over-spending" (from their perspective) by less patient other parties. Nevertheless, it is routinely criticized as an unjustifiable withholding of charitable funds, or even as a form of tax evasion. These criticisms have recently reached new prominence in the United States with the June 2021 introduction of the Accelerating Charitable Efforts ("ACE") Act by Senators Angus King and Charles Grassley, which would impose disbursement requirements on DAFs, effectively requiring their contributors to act less patiently. Due to a lack of literature on dynamic public good provision under time preference heterogeneity, the implications of such a requirement, and of similar proposals to introduce or raise charitable disbursement minima in the United States and elsewhere, have not undergone thorough economic scrutiny.

This paper therefore begins more seriously to explore the implications of time preference heterogeneity for the dynamic provision of public goods. To do so, I use a simple model in which there is a single public good, and the utility produced by spending on it is isoelastic in total flow spending on it. Players with constant but different rates of time preference make decisions about how much to contribute to the good, and how much to invest for future contribution, over an infinite horizon in continuous time.[4]

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ity among individuals and households can be found in Alan and Browning (2010), pp. 1252–3.

[2] Compare US Office of Management and Budget (2019) and HM Treasury (2020), for instance.

[3] As surveyed by Drupp et al. (2018).

[4] I use roughly the notation and framework for dynamic optimization problems and games in

To highlight the implications of time preference heterogeneity *per se*, I assume that the size of each player's total contribution budget is fixed. Contributors decide only the schedule on which to deploy their spending, not the extent to which they will spend on public as opposed to private goods each period. As a result, the model does not resemble the existing literature on dynamic public good provision so much as Bergstrom et al.'s original paper. The relevant change is that I explore what happens when individuals choose their contribution levels for an infinite stream of public goods over which their respective preferences differ—i.e. funding at t, for all $t \geq 0$—*in sequence*, rather than simultaneously. (I sometimes refer to the public good contributors as "philanthropists", and focus on applications to philanthropy, but the mathematical results are applicable to decentralized public good contributors more generally.)

The model, despite its simplicity, allows us to draw some important and broad conclusions about the implications of heterogeneous discounting for public good provision. First, the common discounting assumption is a knife-edge condition: even slight differences in patience, even (indeed especially) by small (i.e. poorly-funded) players, give rise to very different equilibria. In particular, they can give rise to relatively simple and natural equilibria in which spending is "polarized" in the sense given above, with impatient parties exclusively responsible for public good funding before some future date and patient parties exclusively responsible after. Second, this knife-edge condition is payoff-relevant: the equilibria that obtain under heterogeneous discounting can offer payoffs, at least for unusually patient parties, which differ dramatically from the payoffs they achieve under common discounting. Third, and relatedly, disbursement requirements can reduce the payoff to philanthropy dramatically, from a patient perspective.

Besides the literature on public good provision in static settings, and in dynamic settings under homogeneous time preferences, I will now briefly discuss connections to adjacent strands of literature which do not directly concern public good provision but in which the implications of time preference heterogeneity have been explored more extensively.

One such strand concerns the collective allocation of private consumption over time—i.e., under certain preference aggregability assumptions, the discounting behavior of a representative agent—in a population of individuals or lineages with heterogeneous time preferences. One classic observation from this literature is that consumption (Rader, 1981) and/or wealth (Becker, 1980; Ryder, 1985) can, in the limit, become entirely concentrated in the hands of society's most patient members, simply because they consume less and invest more. Another, closely related, as shown in an exchange economy by Gollier and Zeckhauser (2005) (and with varia-

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continuous time presented by Simon and Stinchcombe (1989) and Stinchcombe (2013), and as summarized in Appendix A.

tions elsewhere), is that a representative agent (if one exists) will, under complete markets, exhibit a discount rate that declines with time to that of society's most patient members. In other words, interest rates fall as patient parties lend to their less patient counterparts and command an ever-growing share of the financial market. As we will see, similar dynamics play out among agents spending on public goods, and the effects are often even more extreme.

A second body of relevant research concerns optimal taxation by policymakers more patient than their constituents. Farhi and Werning (2007, 2010) analyze optimal taxation in an intergenerational model where individuals save insufficiently, from the patient social planner's perspective, for their descendants. Household consumption, in these models, is effectively a public good: a good whose provision satisfies the preferences of multiple parties (the policymaker and the household itself) nonexcludably and nonrivally. Similarly, von Below (2012), Belfiori (2017), and Barrage (2018) study optimal carbon taxation and/or investment subsidization in contexts where present production confers both future costs and future benefits (from climate damage and capital accumulation respectively). An important lesson from this literature is that patient policymakers might like to invest resources for future spending, but that to avoid crowding out private investment, it is often optimal for them instead to subsidize private investment and tax private consumption.

Time preference heterogeneity has different implications in the context of optimal taxation than in the context of private spending on public goods, however. The former setting involves an asymmetry in the players' strategy sets: households cannot tax or subsidize policymakers, but policymakers can tax and subsidize households. At least in the absence of political or informational constraints, policymakers endorsing a given time preference rate can often use these tools to implement population-wide behavior that is optimal or near-optimal from the policymaker's perspective. Finally, therefore, a literature has emerged on the implications of time preference heterogeneity among more symmetrically empowered agents, who must *collectively* set discount rate policy for use in public good provision. Gollier and Zeckhauser (2005), as noted above, find conditions under which a group of agents with heterogeneous time preferences give rise to a representative agent whose time preferences are determined by the intertemporal allocation of private consumption, and the corresponding interest rate schedule, that clears the agents' financial contracts. As Heal and Millner (2014) argue, there is a natural sense in which policymakers aiming to set discounting policy for the provision of a public good, while deferring to the time preferences of their constituents, do best to defer to this aggregated discounting schedule.

Finally, an alternative approach to collective social discounting is to consider, in a social-choice-theoretic framework, the decision-making of committees of social planners with different rates of time preference. Such an approach, however, faces varieties of the preference aggregation impossibilities faced in other social choice contexts, as explored in detail by Chambers and Echenique (2018). Millner (2020) proposes a method by which the discounting planners might reach a kind of

consensus, but such proposals are themselves inevitably vulnerable to disagreement. The social choice approach to discounting must also confront the issue of time inconsistency. In particular, Jackson and Yariv (2015) show that any social welfare function used in this setting must be either dictatorial or time-inconsistent, in that future committee meetings will, if they use the same forward-looking social welfare function, decide to revise the plans made by previous meetings—at least if these were made naively, without taking the possibility of future revisions into account. Millner and Heal (2018) therefore examine the collective decision-making of discounting committees aware that they are playing a dynamic game with their future selves. One finding is that attempts to implement "weighted utilitarian" social discounting in such a dynamic game will generally be inefficient. (By contrast, I find that decentralized private actors strategically allocating public good contributions over time can implement efficient, weighted utilitarian social discounting.)

The structure of this paper is as follows.

I begin in §2 with the benchmark scenario in which the good has only a single funder. Barring certain technicalities, the spending schedule obtaining in this case is the same as that which obtains given multiple funders with a shared rate of time preference.

In §3, I explore the interaction between a patient and an impatient philanthropist. I open by discussing, as a second benchmark, what follows when one party is "warm-glow", in the sense of Andreoni (1990) that he is concerned only with his own schedule of contributions to the public good, whereas the other party is "altruistic", in that she cares about the schedule of total contributions. Modeling both parties as altruistic is most standard in the literature on public good contribution games, but I open with a discussion of warm-glow behavior both because it is a widely documented pattern of philanthropic behavior in practice and because it serves as a simple introduction to the nature of the parties' interactions.

I then discuss the interaction between altruistic funders across a sequence of increasingly complex game specifications: one in which the parties simultaneously commit to a spending schedule; one in which the impatient party commits to a spending schedule and the patient party then chooses his best response; and finally one, ultimately of most interest, in which the parties play a dynamic game, choosing strategies that govern their spending rates at each point in time as a function of the joint contribution history up to that time. I find that every game specification produces a unique and Pareto-inefficient equilibrium, except the last, which produces many equilibria of which some are efficient and some are not. I also find that there is a sense in which the impatient party has an advantage in the dynamic game, relative to her status in the game with simultaneous commitment, analogous to the first-mover advantage she enjoys in the second game specification with altruistic funders.

In §4, I highlight the economic significance of time preference heterogeneity by

estimating a funder's willingness to pay to move from spending his budget according to a schedule that is optimal given an alternative time preference rate to spending his budget according to the schedule that is optimal given his own time preference rate, even though doing so may induce an inefficient heterogeneous-discounting equilibrium. I find that, for an atypically patient party (but not for an atypically impatient party), this willingness to pay approaches the entirety of his budget as his budget share—his fraction of the sum of the parties' budgets—goes to zero. That is, when most of the funding for his chosen cause is governed impatiently, the payoff that the patient party attains by acting patiently grows arbitrarily higher than the payoff he attains by spending as he would under common, less patient discounting.

Finally, §5 illustrates conclusions from the earlier sections with what I believe to be plausible parameter values, across examples in which the patient funder's endowment is initially much smaller than, the same size as, and much bigger than the impatient funder's. I focus primarily on the first of these cases. In particular, I consider the extreme possibility that, when taken to its logical conclusion, the game theory of the earlier sections recommends that globally impartial patient philanthropists should, at least under some circumstances, invest their wealth for centuries—until they have amassed a substantial share of global wealth—before disbursing.

§6 concludes.

2 Benchmark 1: Good provision with a single funder

2.1 Model

Let us begin with a model in which an agent is the sole provider of some good over an infinite horizon. Consider, for example, the case of a philanthropist providing non-durable consumption goods for a penniless (but potentially long-lived) individual or lineage.

Let us denote the size of the agent's budget at time $t = 0$ by B. At each moment t, we will assume that the flow utility u achieved by providing the good is an isoelastic function, with inverse elasticity of intertemporal substitution $\eta > 0$, of the rate $X \geq 0$ at which the agent spends. That is,

$$u(X(t)) = \begin{cases} \frac{X(t)^{1-\eta}-1}{1-\eta}, & \eta \neq 1; \\ \ln(X(t)), & \eta = 1. \end{cases} \tag{1}$$

The agent faces a constant instantaneous real interest rate r and a constant instantaneous time preference rate δ. The latter might represent pure time preference, plus the risk of a catastrophe that brings the agent's utility to zero forever after. (This could be the rate of "existential catastrophe", i.e. the risk per unit time that

the world ends or human civilization collapses. Less dramatically, in the case of a philanthropist concerned only with a beneficiary individual or lineage, it could represent this beneficiary's mortality risk.) There does not appear to be a standard term for the quantity we are denoting δ, but we will call it the "time preference rate", reserving the term "discount rate" for the discounting of marginal spending and the term "pure time preference rate" for time preference in a risk-free environment. We need not assume that r or δ is positive.

The agent's problem is then to choose the schedule of spending rates $X(t)$ that maximizes

$$U = \int_0^\infty e^{-\delta t} u(X(t)) dt \tag{2}$$

subject to the constraint

$$\int_0^\infty e^{-rt} X(t) dt \leq B. \tag{3}$$

Proposition 1. *Optimal individual spending schedule*
Suppose an agent has isoelastic utility in spending with inverse elasticity of intertemporal substitution $\eta > 0$, a constant time preference rate δ, and a budget $B > 0$, and suppose she can invest her resources at a constant interest rate r. Then, if $\delta > r(1 - \eta)$, the agent uniquely[5] maximizes discounted utility by following spending schedule

$$X(t) = B \frac{r\eta - r + \delta}{\eta} e^{\frac{r - \delta}{\eta} t}. \tag{4}$$

The payoff to following this spending schedule is

$$\frac{B^{1-\eta}}{1-\eta} \left(\frac{r\eta - r + \delta}{\eta} \right)^{-\eta} - \frac{1}{\delta(1-\eta)}, \quad \eta \neq 1; \tag{5}$$

$$\frac{\delta \ln(B\delta) + r - \delta}{\delta^2}, \quad \eta = 1.$$

If $\delta \leq r(1 - \eta)$, there is no optimal spending schedule.

Proof. See Appendix B.1. $\square$

The above model is motivated in this paper by the scenario in which a philanthropist is the sole provider of some good. So far, it is equivalent to an infinite-horizon consumption-smoothing model under certainty, assuming either (a) no future outside income or (b) complete capital markets. (Note that the assumption of complete capital markets renders this problem the same as the problem one faces with no

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[5]As referenced in §1 (see Fn. 4), I here follow a framework in which continuous time optimization is defined to be the limit of discrete time optimization as time is discretized across an ever finer grid. As explained at more length in Appendix A, this framework allows us to characterize the following spending schedule as the unique optimum, rather than just an optimum unique up to measure-zero deviations.

outside income. Given certainty and complete markets, someone with future income can borrow against her entire income stream, and B can represent current assets plus the present value of future income.)

In the context of a simple consumption-smoothing model, the optimal spending rate is highly sensitive to the discount rate. The patient, that is, should spend slowly. As we can see from (4) at $t = 0$, it is always optimal to spend at proportional rate $\frac{r\eta - r + \delta}{\eta}$. In particular, if $\eta = 1$, the spending rate should equal δ. For instance, philanthropists who are funding idiosyncratic projects with no other present or future funders, who discount future impacts at 0.1% per year, and who are confident that the world (or their philanthropic projects) will not soon be brought to an end, should spend only 0.1% of their budgets per year.

In fact, unless

$$\delta > r(1 - \eta), \tag{6}$$

it is always preferable to delay spending at all times than to begin spending immediately. That is, if (6) does not hold, then given any feasible spending schedule X, we find that for any $s > 0$, the feasible spending schedule $\tilde{X}$ defined by

$$\tilde{X}(t) = \begin{cases} 0, & t < s; \\ e^{rs} X(t - s), & t \geq s \end{cases} \tag{7}$$

satisfies $U(\tilde{X}) > U(X)$. Inequality (6) is thus necessary for an optimal spending schedule to exist; if it is violated, we face Koopmans' (1967) "paradox of the indefinitely postponed splurge". Note that it does so whenever $\eta > 1$, $r > 0$, and $\delta \geq 0$. That is, under the standard assumptions that $r > 0$ and $\eta > 1$, an optimal spending schedule exists even for a fully patient agent.

2.2 The economic importance of time preference

As noted in §1, small donors can currently invest philanthropically-purposed assets in tax-free funds and disburse them at their own pace, but these funds risk soon being subjected to a legal disbursement minimum. To begin to shed light on patient donors' willingness to pay to avoid this requirement, at least in the single-funder model of the current section, we will now calculate a patient actor's willingness to pay for the right to move from a high disbursement rate to a lower, patient-optimal disbursement rate.

Similarly, large philanthropists face a choice between holding their capital in a foundation and holding it privately or in a trust. In the United States, contributions to foundations are tax-exempt, as are the capital gains their assets earn. Foundations must disburse at least 5% of their assets per year, however, effectively requiring them to act impatiently. Trusts are not tax-exempt but are not subject to such a requirement. The willingness-to-pay calculation below can thus inform large patient philanthropists about how significant the tax advantage to a foundation must be to

justify this loss of freedom over the implicit choice of time preference rate (if they place negligible value on marginal tax contributions).

In §4, we will compare these estimates to the economic importance to patient actors of patient behavior in the context of strategic interactions with less patient co-funders.

To begin, let us determine the payoff, for an agent with time preference rate δ, to spending according to some time preference rate $\tilde{\delta} \neq \delta$.

The patient payoff to spending according to $\tilde{\delta}$ can be found by substituting $\tilde{\delta}$ for δ in (4) to get the $\tilde{\delta}$-optimal spending schedule. Then, substitute this schedule as $X(t)$ into (2) to get

$$\int_0^\infty e^{-\delta t} u\left(B\, \frac{r\eta - r + \tilde{\delta}}{\eta}\, e^{\frac{r-\tilde{\delta}}{\eta}t}\right) dt. \tag{8}$$

Observe that this will only be defined if

$$\eta \leq 1 \ \text{ or } \ \tilde{\delta} < r + \delta\frac{\eta}{\eta - 1}. \tag{9}$$

If $\eta > 1$ and $\tilde{\delta}$ is too high, the $\tilde{\delta}$-optimal plan may push the spending rate to 0 quickly enough that, though this produces finite $\tilde{\delta}$-discounted disutility, it produces infinite δ-discounted disutility. Note that $\tilde{\delta} < r + \delta$ is sufficient to avoid this condition.

Proposition 2. *Payoff to spending according a given time preference rate*

Suppose the conditions of Proposition 1 are satisfied for an agent. Then given some $\tilde{\delta} \geq \delta$, if condition (9) is met, the agent's payoff to following the $\tilde{\delta}$-optimal spending schedule is

$$U_\delta(B, \tilde{\delta}) = \begin{cases} \frac{B^{1-\eta}(r\eta - r + \tilde{\delta})^{1-\eta}\eta^\eta}{(1-\eta)(\delta\eta - (r-\tilde{\delta})(1-\eta))} - \frac{1}{\delta(1-\eta)}, & \eta \neq 1; \\[2ex] \frac{\delta\ln(B\tilde{\delta}) + r - \tilde{\delta}}{\delta^2}, & \eta = 1. \end{cases}$$

Proof. Integrate (8) subject to (9). $\qquad\qquad\square$

We can now calculate how much of her budget a patient agent should be willing to give up to move from the $\tilde{\delta}$-optimal to the δ-optimal spending schedule.

Proposition 3. *WTP for acting on time preference*
Suppose the conditions of Proposition 1 are satisfied for an agent, and consider a time preference rate $\tilde{\delta}$ such that condition (9) is met. Then in order to spend her resources as would be optimal given δ as opposed to $\tilde{\delta}$, she is willing to give up the

following fraction of her budget:

$$1 - \frac{r\eta - r + \tilde{\delta}}{r\eta - r + \delta}\left(\frac{r\eta - r + \delta}{r\eta - r + \tilde{\delta} - (\tilde{\delta} - \delta)\eta}\right)^{\frac{1}{1-\eta}}, \ \eta \neq 1;$$

$$1 - \frac{\tilde{\delta}}{\delta}e^{1-\frac{\tilde{\delta}}{\delta}}, \ \eta = 1.$$

Proof. Using the payoff expressions from Proposition 2, set $U_\delta(B, \tilde{\delta}) = U_\delta((1 - w)B, \delta)$ and solve for w. □

Concretely, suppose $r = 5\%$. Then the value achieved by spending according to time preference rate $\tilde{\delta} = 2\%$, by the lights of time preference rate δ, is equal to the value achieved by giving up the following budget-fractions but spending the remaining budget according to time preference rate δ:

| η | WTP given... | | | |
	$\delta = 0.1\%$	$\delta = 0.5\%$	$\delta = 8\%$	$\delta = 40\%$
0.99	$1 - 4.0 \times 10^{-13}$	0.84	0.48	0.87
1	$1 - 1.1 \times 10^{-7}$	0.80	0.47	0.87
1.01	$1 - 1.8 \times 10^{-5}$	0.77	0.46	0.87
1.25	0.58	0.29	0.36	0.82
2	0.14	0.07	0.21	0.71

0.1% is the time preference rate used by Stern (2006) to represent exogenous risk of civilizational collapse, and has since become a standard "patient" time preference rate to use for agents who reject pure time preference. 2% is a common rough estimate of the time preference rate that most households employ.

As we can see, a patient agent can err substantially by spending as would be optimal given a more typical time preference rate. That is, implicitly spending according to discount rate $\delta = 2\%$ is a mistake she should be willing to give up a substantial part of her budget to avoid. Furthermore, this willingness to pay is highly sensitive to the values of η and δ. It is most extreme for low values of η and δ: in this case it is almost her entire budget. Even when $\eta = 2$ and $\delta = 0.5\%$, however, spending as if her time preference rate were 2% is tantamount to a loss of about 7% of her resources.[6]

Furthermore, in this context there is a technical symmetry between the case of an patient funder required to spend patiently and an impatient funder required to spend

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[6]It may be of interest that Mathematica numerically determines the willingness-to-pay expression of Proposition 3 to be decreasing in η across all admissible parameters, not only those in the table above. That is, as is perhaps intuitive, the relative benefit to acting on one's patience appears to be increasing in one's elasticity of intertemporal substitution.

impatiently. As follows relatively directly from the statement of the proposition, as $\delta \downarrow r(1 - \eta)$ or $\delta \uparrow \infty$ (holding $\tilde{\delta}$ fixed), an agent's WTP to act on time preference δ rises to 1, and (given the homotheticity of u) this convergence is independent of budget size. By contrast, as we will see in §4, the interaction of patient and impatient funders introduces strong asymmetries, in which small and atypically patient funders are willing to pay almost everything to act on their true time preferences but small and atypically impatient funders are not.

3 Interaction between patient and impatient funders

3.1 Motivation and framework

The model above allows us to determine the optimal spending policy regarding the provision of a good for which there is only one purchaser. It applies, for instance, to the schedule on which an individual should allocate her private spending, or on which a philanthropist only interested in funding an esoteric project should allocate his spending on that project. When one is a philanthropist providing a public good to which others also contribute (or would contribute absent one's own funding), however, one must consider the ways in which one's own funding affects the behavior of the good's other funders. In particular, when one is a patient philanthropist, one must remember that investment for future spending can induce less patient funders to spend more quickly.

As we will see, intertemporal free-riding and crowd-out concerns can motivate substantially different—and generally even "more patient"—behavior from a patient philanthropist than is optimal in the single-funder context. In particular, a patient philanthropist often does best in the presence of impatient funders to invest *all* his resources, for some period, and then to spend on an exponential schedule resembling the single-funder schedule determined above.

Throughout the results below, we will posit a single impatient party I and a single patient party P, both satisfying the conditions of Proposition 1 (with a common η and r but individual discount rates and budgets). We will denote party i's budget at time t by $B_i(t)$, the total budget by $B(t) \triangleq B_I(t) + B_P(t)$, and i's budget proportion by $b_i(t) \triangleq B_i(t)/B(t)$, for $i \in \{I, P\}$. Terms without time arguments (B_i, B, b_i) will denote their values at $t = 0$. Finally, we will denote party i's time preference rate by δ_i, for $i \in \{I, P\}$. We will assume that the time preference rates satisfy conditions (6) and (9), with δ_P as δ and δ_I as $\tilde{\delta}$.[7]

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[7]If P rejects *pure* time preference altogether, $\delta_I - \delta_P$ equals the portion of the impatient time preference rate consisting of pure time preference, as distinct from discounting for e.g. risk of civilizational collapse.

We will introduce a more precise framework when necessary, at the beginning of §3.4. For now, let us omit some technicalities and say only that at every moment $t \geq 0$, the players observe the spending history $\{(X_I(s), X_P(s))\}_{s<t}$ and independently choose spending rates $X_I(t)$ and $X_P(t)$ respectively.

Throughout the following sections, it will be helpful to define:

$$\alpha_i \triangleq \frac{r\eta - r + \delta_i}{\eta}, \quad i = I, P;$$

$$\gamma \triangleq \begin{cases} \left(\frac{\alpha_P + \delta_I - \delta_P}{\alpha_I}\right)^{\frac{1}{1-\eta}}, & \eta \neq 1; \\ \\ e^{\frac{\delta_P - \delta_I}{\delta_I}}, & \eta = 1. \end{cases}$$

α_i denotes the optimal spending rate of an agent of type i, as a proportion of his budget per unit time, when this agent is the only provider of a good. Note that α_I, α_P, and γ are all positive under all admissible parameters, and that $\gamma < 1$.

3.2 Benchmark 2: Interaction between warm-glow and altruistic funders

Following Andreoni (1990), let us define two forms that a funder's utility function might take.

Definition 1. *Funder i is **altruistic** if her utility function is given by*

$$U_i = \int_0^\infty e^{-\delta_i t} u(X_I(t) + X_P(t)) dt,$$

where $u(\cdot)$ is an isoelastic function parametrized by η, as before.[8]

Definition 2. *Funder i is **warm-glow** if her utility function is given by*

$$U_i = \int_0^\infty e^{-\delta_i t} u(X_i(t)) dt,$$

with $X_i(t)$, rather than $X_I(t) + X_P(t)$, as the argument of her flow utility function.

That is, the funder is defined to be altruistic if she is concerned with the extent to which the public good is funded, regardless of who funds it. The funder is defined to be warm-glow if she is concerned with the extent to which she herself funds the public good, e.g. because she derives some benefit from being seen as charitable or because she enjoys the act of giving. As Andreoni (1990) documents, both types of

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[8]If $\eta \geq 1$ and both parties spend 0 at any t, we can without complications define total utility for both parties to equal $-\infty$, as does Laibson (1994) in a similar context.

philanthropy (and types along the spectrum that joins these extremes) are widely observed.

Let us therefore begin by determining optimal altruistic spending behavior by j in the presence of a warm-glow funder i exhibiting a different rate of time preference. The game the funders play is simple: at each time s, i chooses the forward-looking spending schedule $X_i(t)$ ($t \geq s$) that maximizes her forward-looking utility. Given this anticpated schedule of spending by i, j simultaneously chooses the forward-looking spending schedule $X_j(t)$ ($t \geq s$) that maximizes his forward-looking utility.

Proposition 4. *Altruistic spending given a warm-glow co-funder*
The dynamic public good contribution game described above has a unique equilibrium. In this equilibrium, if I is warm-glow and P is altruistic, P follows spending schedule

$$X_P(t) = \begin{cases} 0, & t < t^*; \\ \left(B_I e^{(r-\alpha_I)t^*} + B_P e^{rt^*}\right)\alpha_P e^{(r-\alpha_P)(t-t^*)} - B_I \alpha_I e^{(r-\alpha_I)t}, & t \geq t^*, \end{cases} \tag{10}$$

where

$$t^* = \max\left(0, \ln\left(\frac{b_I}{b_P}\frac{\alpha_I - \alpha_P}{\alpha_P}\right)\Big/\alpha_I\right). \tag{11}$$

If P is warm-glow and I is altruistic, I follows spending schedule

$$X_I(t) = \begin{cases} B_P \alpha_P \left(e^{(\alpha_I - \alpha_P)t^* + (r-\alpha_I)t} - e^{(r-\alpha_P)t}\right), & t < t^*; \\ 0 & t \geq t^*, \end{cases} \tag{12}$$

where t^* uniquely satisfies

$$\frac{\alpha_I}{b_P}e^{\alpha_P t^*} - \alpha_P e^{\alpha_I t^*} = \alpha_I - \alpha_P \tag{13}$$

(but has no closed-form solution).

Proof. See Appendix B.2. □

Because the preferences of a warm-glow funder are the same as those of a single funder, the warm-glow spending schedule is always given by Proposition 1.

As we can see, given a warm-glow impatient party, an altruistic patient party does best to invest *all* his resources as long as his share of total resources is sufficiently low: in particular, as long as

$$b_P(t) < \frac{\alpha_I - \alpha_P}{\alpha_I}. \tag{14}$$

The intuition is straightforward. If the impatient party controls a large enough share of total resources, the impatient-optimal spending rate at which to spend her own budget may be higher not just than the patient-optimal rate at which to spend her

budget, but than the patient-optimal rate at which to spend the collective budget. If so, any spending by the patient party would, on his view, increase the extent to which they are collectively overspending. He should only begin spending once the impatient party's share of the collective budget has shrunk enough that even spending it impatient-optimally constitutes underspending the collective budget, from the patient perspective.

Likewise, given a warm-glow patient party, an altruistic impatient party does best to spend *all* her resources in finite time, "topping up" P's spending only so long as P underspends the collective budget from an impatient perspective.

Thus, in the context of altruistic spending on public goods, optimal patient and impatient behavior can differ by even more than they differ in the context of private spending.

3.3   Benchmark 3: Static interaction among altruistic funders

If both funders are altruistic, their spending problems take the form of a dynamic game. There is already a substantial literature on dynamic public goods contribution games, but none of it yet appears to have considered the implications of differences in time preference. The framework used here is designed to introduce such differences. In fact it isolates the effects of differences in time preference by positing that they are the funders' *only* preference differences; funders do not have the opportunity to spend on private goods which only they value, as they are typically assumed to have. We will therefore hold fixed the size of the budget each party contributes to the public good.

Before exploring the dynamic game in question, however, let us consider two closely related static games.

First, let us suppose that each player simultaneously commits at time 0 to a schedule on which he or she will spend his or her budget over the horizon from $t = 0$ to ∞. The players here can commit only to absolute spending schedules, not to history-dependent spending policies.

Proposition 5. *Existence and uniqueness of Nash equilibrium in the simultaneous-move game*
Suppose that, at $t = 0$, each funder i sets the entire spending schedule $X_i(t)$ simultaneously. This game has a unique Nash equilibrium, in which

$$X_I^*(t) = \begin{cases} (B_I \alpha_I + B_P \alpha_P)e^{(r-\alpha_I)t}, & t < t^*; \\ 0, & t \geq t^* \end{cases} \tag{15}$$

and

$$X_P^*(t) = \begin{cases} 0, & t < t^*; \\ B_P\alpha_P\left(1 + \frac{B_I\alpha_I}{B_P\alpha_P}\right)^{\frac{\alpha_P}{\alpha_I}} e^{(r-\alpha_P)t}, & t \geq t^*, \end{cases} \tag{16}$$

where

$$t^* = \ln\left(1 + \frac{B_I\alpha_I}{B_P\alpha_P}\right)\Big/\alpha_I. \tag{17}$$

Proof. See Appendix B.3. □

This is essentially a special case of the static public good contribution game analyzed by Bergstrom et al. (1986), but with a continuum of public goods: spending at each t, for $t \in [0, \infty)$. As in the analogous Bergstrom et al. case, we here find that each good is provided by exactly one funder, and a funder always provides a good when he is also providing a good for which he cares relatively less. That is, since P cares relatively more than I about spending at t the later t is, there is a threshold time t^* such that I is the sole funder before t^* and P is the sole funder after.

Furthermore, the spending rate is continuous at t^*; $\lim_{t \uparrow t^*} X_I^*(t) = X_P^*(t^*)$. If the spending rate rose discontinuously at t^*, P would do better to reallocate some spending from $t^* + \epsilon$ to $t^* - \epsilon$ for some sufficiently small $\epsilon > 0$. Likewise, if the spending rate fell discontinuously, I would do better to reallocate marginal spending forward.

Next, let us suppose that I is the "Stackelberg leader". That is, let us suppose that, at $t = 0$, I sets a feasible spending schedule $X_I(t)$, and P sets a feasible spending schedule $X_P(t)$ in response. As in the simultaneous-move game above, both spending schedules are here set in their entirety at $t = 0$; the players cannot set history-dependent spending policies.

Proposition 6. *Existence and uniqueness of subgame-perfect equilibrium in the Stackelberg game*
Suppose that, at $t = 0$, I sets spending schedule $X_I(t)$, and P sets spending schedule $X_P(t)$ in response. This game has a unique[9] subgame-perfect equilibrium across the two periods, which induces spending schedules

$$X_I^*(t) = \begin{cases} B_I\frac{Z}{Z-1}\alpha_I e^{(r-\alpha_I)t}, & t < t^*; \\ 0, & t \geq t^* \end{cases} \tag{18}$$

---

[9]Unlike the other results in this paper, this proposition is proven on the assumption that the parties can directly choose spending schedules defined over the whole real line, rather than one in which permissible continuous-time spending rates must take the limit of a sequence of discrete-time spending rates (see Appendix A). The equilibrium is thus found to be unique only up to measure-zero deviations; spending schedules differing from the below on measure-zero sets of times may also obtain in equilibrium.

and

$$X_P^*(t) = \begin{cases} 0, & t < t^*; \\ B_P \, Z^{\frac{\alpha_P}{\alpha_I}} \, \alpha_P \, e^{(r-\alpha_P)t}, & t \geq t^* \end{cases} \tag{19}$$

respectively, where

$$t^* = \ln(Z)/\alpha_I \tag{20}$$

and

$$Z \triangleq 1 + \frac{B_I \alpha_I}{B_P \alpha_P} \, \gamma. \tag{21}$$

Proof. See Appendix B.4. $\square$

Since $\gamma < 1$, the regime-switching time t^* occurs earlier in the Stackelberg case than in the case where the funders set their spending plans simultaneously. Furthermore, recall that in the simultaneous-move case, spending is continuous at t^*. Here, the spending rate falls discontinuously at t^*, as I allocates budget B_I over a shorter time interval and P stretches B_P over an infinite horizon beginning earlier.

3.4 Dynamic interaction among altruistic funders

We will now explore the funders' interaction in a dynamic setting. To formalize this game, we will use a simplification of the framework of Stinchcombe (2013), summarized in Appendix A.1–A.3. For expository purposes, we will here summarize the notation.

We will define a complete (spending) history as an assignment of a spending rate $X_{i,t}$ to each player i for each $t \geq 0$. It will be denoted by

$$X \triangleq \{(X_{I,t}, X_{P,t})\}_{t=0}^{\infty}. \tag{22}$$

An "open partial history" truncated just before some t will be denoted $X_{|t}$. In a slight abuse of notation, we will denote total spending at t by X_t.

A feasible history X is one whose spending schedules are integrable and feasible for each player: that is, one in which

$$\int_0^{\infty} e^{-rt} X_{i,t} dt \leq B_i \tag{23}$$

for each i.

We will denote the set of decision nodes by $\mathbb{D}$. Note that this is precisely the set of open partial feasible histories: $\mathbb{D} \triangleq \{X_{|t}\} : X$ is feasible, $t \geq 0$.

A strategy σ_i for player i is a function from nodes to spending rates, i.e. $\sigma_i : \mathbb{D} \to \mathbb{R}_{\geq 0}$. Player i's strategy set will be denoted Σ_i, strategy profiles will be denoted $\sigma \triangleq (\sigma_I, \sigma_P)$, and the set of strategy profiles will be denoted $\Sigma \triangleq \Sigma_I \times \Sigma_P$.

We will denote i's budget at time t given feasible history X—i.e. at node $X_{|t}$—by $B_i(X_{|t})$. Likewise, the total budget will be denoted $B(X_{|t})$ and budget proportions $b_i(X_{|t})$.

We will denote by $\chi(\sigma)$ the history induced by strategy profile σ.[10] Given $X_{|t} \in \mathbb{D}$, if the players subsequently adopt strategy profile σ, we will denote the resulting history by $\chi(X_{|t}, \sigma)$.

A strategy profile σ^* is a subgame-perfect equilibrium[11] if, for all $X_{|t} \in \mathbb{D}$,

$$\int_t^\infty e^{-\delta_i s} u(\chi(\sigma^*)_s) ds \geq \int_t^\infty e^{-\delta_i s} u(\chi(X_{|t}, (\sigma_i, \sigma_j^*))_s) ds \ \ \forall \sigma_i \in \Sigma_i \qquad (24)$$

for both players i (where j denotes the other player).

Definition 3. *A **defection profile** of the dynamic game above is a strategy profile σ^D in which*

$$B_I(X_{|t}) > 0 \iff \sigma_P^*(X_{|t}) = 0 \ \ \forall X_{|t} \in \mathbb{D}. \qquad (25)$$

In other words, a defection profile σ^D is one which maintains a history $\chi(\sigma^D)$ in which the patient party does not fund the public good until the impatient party has disbursed all her resources—i.e. until

$$t^* \triangleq \min(\{t : B_I(\chi(\sigma^*)_{|t}) = 0\}). \qquad (26)$$

As we will now see, the game has infinitely many subgame-perfect equilibria, but exactly one "defection equilibrium". That is, one subgame-perfect equilibrium is a defection profile.

Theorem 1. ***Existence, uniqueness, and Stackelberg-equivalence of defection equilibrium***
The dynamic game above exhibits a unique defection equilibrium σ^D. The defection equilibrium induces spending rates $X_{i,t}(\sigma^D) = X_i^*(t)$ for each i, where the $X_i^*(t)$ are defined as in Proposition 6.

Proof. See Appendix B.5. □

Note that the defection equilibrium is Markov perfect, taking (B_I, B_P) as the state variable.

Let us call
$$X_t^D \triangleq \chi(\sigma^D)_{I,t} + \chi(\sigma^D)_{P,t} \qquad (27)$$

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[10]The result that that a strategy profile induces a determinate history, in continuous time, requires the more precise framework; see Appendix A.3.

[11]Roughly; see Appendix A for a discussion of infinitesimal deviations.

the "defection schedule". Like the Stackelberg schedule, it follows an impatient-optimal spending schedule for $t \in [0, t^*)$ and a patient-optimal spending schedule for $t \in [t^*, \infty)$. Because it is equal to the Stackelberg schedule, we know that I strictly prefers it to the Nash schedule, which is also feasible for I in the Stackelberg setting. By analogous reasoning it is strictly dispreferred to the Nash schedule by P, who would prefer to slow I's spending with a later, rather than an earlier, regime-switching time t^*.

In effect, it appears that in a public good contribution game between funders whose preference-differences consist of differences in time preference, the impatient funder has a first-mover advantage, allowing her to do better than her counterpart in a static public good contribution game between funders with preference-differences over a continuum of single-period goods.

Except in the trivial cases where $b_I = 1$ or 0 (so that $t^* = 0$ or ∞, respectively), X^D is inefficient. The impatient party is indifferent regarding marginal reallocations of resources from $t_1 < t^*$ to $t_2 \in (t_1, t^*]$, whereas the patient party strictly prefers them. Likewise the patient party is indifferent regarding marginal reallocations of resources from $s_2 > t^*$ to $s_1 \in (t^*, s_2)$, whereas the impatient party strictly prefers them. If the parties could contract, therefore, they could achieve a Pareto improvement by shifting spending toward t^* from both sides.

However, an enforceable contract is not necessary to achieve efficiency.

Proposition 7. *Efficient equilibria*
A spending schedule is efficient iff it maximizes discounted utility using declining time preference rate

$$\delta(t) = \frac{a\delta_I e^{-\delta_I t} + (1-a)\delta_P e^{-\delta_P t}}{ae^{-\delta_I t} + (1-a)e^{-\delta_P t}}$$

for some $a \in [0, 1]$. Furthermore, every efficient Pareto improvment to the defection schedule can be obtained in a subgame perfect equilibrium.

Proof. See Appendix B.6.   □

Thus, the common result that continuous-time public good contribution games with perfect monitoring admit efficient subgame-perfect equilibria also applies to the current case, in which preference differences consist of time preference differences.

Finally, as we can see, an efficient spending schedule uses a time preference rate that declines from $a\delta_I + (1-a)\delta_P$ at $t = 0$ to δ_P as $t \to \infty$, as long as $a < 1$. In doing so, it follows the same path as the declining discount rate that is optimal under discount rate uncertainty, for an agent placing probability a on the validity of discount rate δ_I and probability $1 - a$ on that of δ_P. It is also optimal from the perspective of a "weighted utilitarian" social planner intending to place weight a on I's preferences and $1 - a$ on P's, at time 0.

As noted in §1, Jackson and Yariv (2015) and Millner and Heal (2018) find that such a planner—or analogously, a committee of planners fraction a of whom are of the impatient type, who wish to aggregate their preferences over spending schedules according to the utilitarian social welfare function—will be unable, without commitment, to implement this optimal spending schedule. Proposition 7 tells us that, by devolving public good provision to private parties, such a spending schedule *is* implementable, as long as the allocation of the initial budget to each party is within the range such that the desired efficient schedule is a Pareto improvement on the defection schedule. (This may require an "over-endowment" of the patient party, due to the "first-mover advantage" the impatient party enjoys under defection.) These devolved budgets must be earmarked for public good provision, but free from requirements on the disbursement schedule, as private trusts typically are in both respects.

4   Costs of common discounting

4.1   Payoffs without common discounting

Spending the collective budget according to a fixed rate of time preference $\delta \in (\delta_P, \delta_I)$ is not Pareto-efficient. Proposition 7 thus establishes that there are equilibria of the dynamic public good provision game which are preferred by both parties to spending collectively under a constant time preference rate. As in §2.2, let us now explore the economic importance of this preference by determining "how much" a patient or impatient altruist errs by agreeing to spend the collective budget according to an intermediate time preference rate. As we will see, unlike in §2.2, this error can be arbitrarily large, even for an arbitrarily small discount rate divergence.

To begin, from Proposition 2 at $\delta = \delta_i$, alternative rate $\tilde{\delta}$, and $B = B_I + B_P$ (still denoted B), we have i's payoff (if altruistic) in the event that the collective budget is spent $\tilde{\delta}$-optimally. As in that context, let us denote this payoff $U_{\delta_i}(B, \tilde{\delta})$. We will now compute i's payoff in equilibrium in the presence of warm-glow and altruistic impatient co-funders respectively, assuming that the co-funder does not share i's rate of time preference.

Proposition 8. *Altruistic payoff given a warm-glow co-funder*
If I is warm-glow and P is altruistic, P attains payoff

$$\frac{B^{1-\eta}}{1-\eta}\alpha_P^{-\eta} - \frac{1}{\delta_P(1-\eta)}, \ \eta \neq 1;$$

$$\frac{\ln(B\delta_P) - 1}{\delta_P} + \frac{r}{\delta_P^2}, \ \eta = 1$$

if $b_P \geq (\alpha_I - \alpha_P)/\alpha_I$, and

$$\frac{B_I^{1-\eta}}{1-\eta}\alpha_I^{1-\eta}\left[\left(\frac{1}{\delta_I-\delta_P-\alpha_I}+\frac{1}{\alpha_P}\right)\left(\frac{B_I}{B_P}\frac{\alpha_I-\alpha_P}{\alpha_P}\right)^{\frac{\delta_I-\delta_P-\alpha_I}{\alpha_I}}-\frac{1}{\delta_I-\delta_P-\alpha_I}\right]-\frac{1}{\delta_P(1-\eta)}, \ \eta \neq 1; \quad (28)$$

$$\frac{1}{\delta_P}\left[\ln(B_I\delta_I)+\frac{r-\delta_I}{\delta_P}+\left(\frac{B_I}{B_P}\right)^{-\frac{\delta_P}{\delta_I}}\left(\frac{\delta_I-\delta_P}{\delta_P}\right)^{\frac{\delta_I-\delta_P}{\delta_I}}\right], \qquad \eta = 1$$

if $b_P < (\alpha_I - \alpha_P)/\alpha_I$.

   If P is warm-glow and I is altruistic, I's payoff is bounded above by

$$U_{\delta_I}(B_P, \delta_P) + B_I(B_P\alpha_P)^{-\eta}, \tag{29}$$

which it approaches as $b_I \to 0$.

Proof. See Appendix B.7.   □

   In the presence of an altruistic impatient funder, we have seen that there are multiple equilibria, offering the parties different payoffs. Without an equilibrium selection argument, therefore, the payoff to each party is indeterminate. Because of the defection equilibrium's simplicity and Markov perfection, however, we may loosely take the defection payoffs to be natural lower bounds on the payoffs to expect to accrue each party.

Proposition 9. *Defection payoff given an altruistic co-funder*
In the presence of an altruistic impatient funder, if the funders engage in the defection equilibrum or a Pareto-superior equilibrium, a patient funder attains a payoff of at least

$$\frac{B_P^{1-\eta}}{1-\eta}\,\alpha_P^{-\eta}\,\frac{\alpha_I\alpha_P\eta Z^{1-\eta}+(\delta_I-\delta_P)^2(1-\eta)Z^{-\frac{\alpha_P}{\alpha_I}\eta}}{\alpha_I\alpha_P\eta+(\delta_I-\delta_P)^2(1-\eta)}-\frac{1}{\delta_P(1-\eta)}, \ \eta \neq 1; \quad (30)$$

$$\frac{1}{\delta_P}\left[\frac{(\delta_I-\delta_P)^2}{\delta_I\delta_P}Z^{-\frac{\delta_P}{\delta_I}}+\ln(B_P\delta_PZ)+\frac{\delta_I-\delta_P}{\delta_I}+\frac{r-\delta_I}{\delta_P}\right], \ \eta = 1,$$

and an impatient funder a payoff of at least

$$\frac{B_P^{1-\eta}}{1-\eta}\,\alpha_P^{-\eta}\,Z^{-\eta}\left(\frac{B_I}{B_P}\gamma^\eta+\frac{\alpha_P}{\alpha_P+\delta_I-\delta_P}\right)-\frac{1}{\delta_I(1-\eta)}, \ \eta \neq 1; \quad (31)$$

$$\frac{1}{\delta_I}\left[\ln\left(B_P\delta_PZ\right)+\frac{r-\delta_P}{\delta_I}\right], \ \eta = 1,$$

where Z is defined as in Proposition 6.

Proof. Integrate δ_i-discounted utility given the spending rates from Theorem 1. That is, calculate

$$\int_0^{t^*} e^{-\delta_i t}u\left(B_I\frac{Z}{Z-1}\alpha_I e^{(r-\alpha_I)t}\right)dt + \int_{t^*}^\infty e^{-\delta_i t}u\left(B_P\,Z^{\frac{\alpha_P}{\alpha_I}}\,\alpha_P\,e^{(r-\alpha_P)t}\right)dt,$$

for $i = I, P$, where t^* is defined as in Proposition 6.   □

4.2 Defining a compromise rate

In §4.3, we will compare the extent to which P values the right to spend patiently in a decentralized equilibrium, rather than join I in collective spending under time preference rate $\delta \in (\delta_P, \delta_I]$, with the extent to which I correspondingly values the right to spend impatiently. Though the central implication of the results will not depend on the details of the chosen collective spending rate, it may be instructive to compare the parties' payoffs under a decentralized equilibrium to those that obtain under a particular "compromise rate", motivated as follows.

When a public policy offers intertemporal costs and benefits, its welfare analysis requires a method of comparing the value of costs and benefits at different times. A standard procedure is to discount future flow utility using a constant time preference rate δ inferred from observed interest rates, growth rates, and inverse elasticities of intertemporal substitution using the Ramsey Formula:

$$r = \delta + \eta g \quad \Longrightarrow \quad \delta = r - \eta g \tag{32}$$

(see e.g. Freeman et al., 2018).

In a world without risk, in which r is exogenous and constant,[12] and in which agents exhibit a common isoelastic flow utility function parametrized by η and a common time preference rate δ, the economic growth rate g is determined by individuals' investment decisions to equal $(r - \delta)/\eta$. (This follows immediately from Proposition 1.) The common time preference rate then does indeed equal $r - \eta g$.

In this world, private parties will disburse philanthropic funds at the δ-optimal rate of $(r\eta - r + \delta)/\eta$, as we have seen. Thus, no policy imposing philanthropic disbursement requirements can improve on the privately chosen disbursement rate, from the perspective of a policymaker engaging in the welfare analysis above.

Suppose however that some households exhibit time preference rate δ_P and some exhibit $\delta_I > \delta_P$, and that these households begin at $t = 0$ with assets collectively totaling B_P^H and B_I^H respectively. Because output at t equals rB_t, the observed instantaneous GDP growth rate will equal the growth rate of collective wealth. This will in turn equal the interest rate minus the proportional collective spending rate:

$$g = r - \frac{B_P^H \alpha_P + B_I^H \alpha_I}{B_P^H + B_I^H} = \frac{B_P^H \frac{r-\delta_P}{\eta} + B_I^H \frac{r-\delta_I}{\eta}}{B_P^H + B_I^H} \tag{33}$$

(ignoring the spending behavior of philanthropic funders, due to their relatively small size). The Ramsey-calibrated aggregate time preference rate is then

$$\delta_C \triangleq r - \eta g = \frac{B_P^H \delta_P + B_I^H \delta_I}{B_P^H + B_I^H} = b_P^H \delta_P + b_I^H \delta_I. \tag{34}$$

---

[12]Note that, by assuming that r is exogenous and by positing capital as the only factor of production, we are implicitly assuming an AK economy. The above relationship between δ and observed variables, however, holds under much more general conditions.

More generally, then, we may define $b_P \delta_P + b_I \delta_I$ to be a natural "compromise" time preference rate among agents P and I with budget shares b_P and b_I. It is the rate we may expect to govern Ramsey-calibrated public policy, if the budget-weighted distribution of time preference rates among these agents does not differ from that among the population at large.

Under time preference rate δ_C, the optimal proportional spending rate—or in a philanthropic context, the optimal disbursement rate—equals the constant

$$\alpha_C \triangleq \frac{r\eta - r + \delta_C}{\eta}, \tag{35}$$

by Proposition 1. Under time preference heterogeneity, however, private parties I and P will *not* indefinitely disburse philanthropic funds at rate α_C, regardless of whether they are both warm-glow, one is warm-glow and one is altruistic, or both are altruistic and they engage in the defection equilibrium or an efficient equilibrium. In particular, in all cases, their collective disbursement rate will ultimately fall exactly or asymptotically to $\alpha_P < \alpha_C$. For a policymaker, therefore, taking the incorrect assumption of time preference homogeneity seriously—or taking the time preference rate implied by the market interest rate as a quasi-democratic aggregate of citizens' time preference rates, and seeking to set optimal policy on the basis of this aggregate rate—might motivate imposing a binding, long-term philanthropic disbursement requirement of rate α_C.

In fact, this is roughly what is done. Foundations in the United States are required to disburse their funds at a rate of at least 5% per year: the optimal disbursement rate given a time preference rate of 2% (as is commonly calibrated to the population at large), an interest rate of 7%, and an observed economic growth rate of approximately 2%. No disbursement maximum is imposed, presumably in part because it is not needed: in the unregulated environment, aggregate proportional disbursement falls toward the patient-optimal rate.

This evaluative framework also shapes the policy discussion around donor-advised funds (DAFs). DAFs are investment vehicles for small donors which are currently both tax-exempt and (at the individual level) exempt from disbursement requirements. As noted in §1, however, proposals to eliminate these tax exemptions or impose disbursement requirements have recently become politically popular in the United States. Andreoni (2018) uses the reasoning outlined above to find that DAFs spend "too slowly", and thus recommends eliminating their tax exemption. Again, this recommendation implicitly relies on the questionable practice of normatively evaluating the social benefits of every donor's spending schedule according to the pooled discount rate implied by the prevailing interest rate.

The economic significance of the implications of disbursement requirements will be discussed further in §5. Here, we will simply note a stark asymmetry between the costs of enforced "compromise"—or, in some sense, of an imposed time preference

homogeneity assumption—to patient and impatient public good providers respectively.

4.3   Asymmetric WTP to avoid common discounting

Define "patient behavior" to mean altruistic patient-optimal private spending in the face of a warm-glow impatient funder, or engagement in the defection equilibrium or a Pareto-superior equilibrium in the face of an altruistic impatient funder. Define "impatient behavior" likewise.

Theorem 2. *Unbounded WTP for patient behavior*
Given an impatient funder of either type, an altruistic patient funder's willingness to pay to engage in patient behavior rather than collective disbursement at rate α_C, as a proportion of his budget, approaches 1 as $b_P \to 0$.

Proof. See Appendix B.8.   □

Corollary 2.1. *Given an impatient funder of either type, an altruistic patient funder's willingness to pay to engage in patient behavior rather than collective disbursement at rate α_I, as a proportion of his budget, approaches 1 as $b_P \to 0$.*

Corollary 2.2. *Given an impatient funder of either type, an altruistic patient funder's willingness to pay to engage in patient behavior rather than face a disbursement minimum of α_C approaches 1 as $b_P \to 0$.*

These corollaries follow from the observations that P strictly prefers collective disbursement at rate α_C both to collective disbursement at rate α_I and to a one-sided disbursement requirement of α_C, which would raise his own disbursement rate without binding on I.

We saw in §3.2–3.4 that the presence of an impatient co-funder, warm-glow or altruistic, should often motivate a patient philanthropist to spend even more slowly than he would if he were the only funder of a given public good. Here we see that if the patient philanthropist spends at a less patient rate, his "loss" (in willingness-to-pay terms) can be much larger with the presence of an impatient co-funder than without—even when spending at the less patient rate comes with the benefit of inducing a much larger impatient co-funder to spend at a more patient "compromise rate". Finally, we see that as the size of the impatient budget increases relative to his own, his loss from spending impatiently grows *arbitrarily* large. That is, regardless of the other parameters, for sufficiently small b_P he is willing to lose approximately his entire budget, and all influence over the spending rate of the impatient budget, in exchange for the right to spend his remaining pittance patiently.

This finding underscores the potentially extreme undesirability of disbursement requirements from a patient perspective, first discussed in §2.2. It also highlights the value that relatively small patient donors may find in recognizing the implications of

their patience for their optimal giving schedules. Small patient donors who hope to contribute to the funding of public goods primarily funded by larger and less patient actors have the most to gain—indeed, in proportional terms, have arbitrarily much to gain—by putting their patience into practice.

By contrast:

Theorem 3. *Bounded WTP for impatient behavior*
Given a warm-glow patient funder, an altruistic impatient funder's willingness to pay to engage in impatient private spending rather than collective spending at rate α_P, as a proportion of her budget, is uniformly bounded below 1 across $b_I \in (0,1)$.

Given an altruistic patient funder, an altruistic impatient funder's willingness to pay to engage in the defection equilibrum rather than collective spending at rate α_P, as a proportion of her budget, equals

$$1 - \frac{\alpha_P}{\alpha_I \gamma}, \tag{36}$$

independent of b_I. Her proportional willingness to pay to engage in the defection equilibrium or a Pareto-superior equilibrium is thus bounded below 1 across $b_I \in (0,1)$.

Proof. See Appendix B.9. □

Corollary 3.1. *Given a patient funder of either type, an altruistic impatient funder's willingness to pay to engage in impatient behavior rather than collective disbursement at rate α_C, as a proportion of her budget, is bounded below 1 across $b_I \in (0,1)$.*

Note that Corollary 2.1 and Theorem 3 demonstrate an important asymmetry that does not depend on the definition of a compromise rate: I's willingness to pay to avoid δ_P-optimal disbursement is bounded, whereas P's willingness to pay to avoid δ_I-optimal disbursement is not. Note also that these results obtain despite the fact that the payoff bounds to each party, in the case that both are altruistic, are given by the defection equilibrium, whose spending schedule is "biased in I's favor" relative to the funding allocation that would obtain in equilibrium among these funders in the analogous static public good contribution game.

5 Applications

As shown by Corollary 2.1, patient parties' gain to spending "strategically", relative to their payoff from spending impatient-optimally, is greatest when the public good under consideration is provided primarily by the impatient parties. In the examples

below, instead of using WTPs explicitly (for which we do not in general have closed-form representations), we will characterize lower bounds on the economic importance of strategic patient behavior, for P, by the payoff ratios

$$\frac{U_P^D - U_P^0}{U_P^I - U_P^0} \quad \text{and} \quad \frac{U_P^{WG} - U_P^0}{U_P^I - U_P^0}, \tag{37}$$

where U_P^0 denotes P's payoff if he spends nothing, leaving I alone to fund the public good; U_P^I denotes P's payoff if he spends δ_I-optimally alongside I; U_P^D denotes P's payoff from engaging in the defection equilibrium; and U_P^{WG} denotes P's payoff from best-responding a warm-glow I. The payoff ratios thus represent "how many times more good P does", by his lights, by best-responding to an altruistic impatient party playing the defection strategy, or respectively a warm-glow impatient party, than by spending impatiently.

To quantify the most extreme possible implications of Corollary 2.1 in practice, let us begin by considering the (very broad) public good with the largest quantity of impatient funding allocated to its provision: global human consumption as a whole.

Note that individual consumption is a public good in a philanthropic setting because it nonexcludably and nonrivalrously satisfies the preferences of two parties: the consumers and the philanthropists who care about them. Given impartial philanthropists who do not restrict their concerns to particular consumption classes (such as medicine or the arts) or to people of a single region, *all* consumption is a public good.

Note also that there is extensive evidence that people tend to employ lower time preference rates when making intertemporal consumption decisions for others than for themselves; see Frietas-Groff (2020) for a recent experimental approach and review of the literature. It is thus reasonable to suppose that philanthropic spending on others' consumption is—and, as patient philanthropists are informed of relevant economic logic, will increasingly be—governed by game-theoretic considerations along the lines presented here.

To set b_P as low as possible, we will assume that the entirety of global wealth is currently purposed to funding human consumption, with everyone exhibiting time preference rate $\delta_I = 2\%$, except for a small community of patient philanthropists with unusual moral commitments to the welfare of future generations, who exhibit time preference rate $\delta_P = 0.5\%$.[13] We will assume an interest rate of $r = 5\%$. We may thus

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[13]This rate is chosen to roughly equal the of annual probability of an expropriation or civilizational collapse that renders philanthropic accumulation worthless, on the basis of estimates by Arbesman (2011) and Sandberg (2019) on the historical longevities of political institutions, and unpublished data compiled by Max Negele on the historical longevities of European universities and Catholic religious institutes. Note that this rate is substantially higher than Stern's (2006) oft-used 0.1% exogenous global hazard rate, and that using the lower rate of 0.1% would increase the calculated payoff gain to patient strategic behavior.

very roughly estimate global wealth, in the relevant sense, to be twenty times gross world product, or $1.8Q.[14] Patient philanthropists' budgets probably cannot total less than $25B, as this is roughly the net present value of the philanthropic budgets of the current Effective Altruism community, a community of philanthropists with explicit commitments to impartiality, including temporal impartiality.[15] So, again for purposes of exploring the most extreme end of the range of possible implications, let $B_I = \$(1.8Q - 25B)$ and $B_P = \$25B$. Finally, let us assume that $\eta = 1.25$, to correspond roughly to (low) estimates of individuals' inverse elasticies of intertemporal substitution with respect to their own consumption.

Under these parameters, if the parties engage in the defection equilibrium, the patient invest exclusively for 437 years—at which point the impatient have disbursed their entire budgets—and subsequently implement the patient-optimal spending schedule. The patient payoff ratio is 291.

In practice, however, for numerous reasons, we may consider it unrealistic to suppose that an impatient world will fully respond to patient philanthropists' long-term investment strategy by making plans to spend all their (and their heirs') wealth within the next few centuries, leaving the subsequent future entirely in the hands of the heirs of the patient philanthropists. At the other end of this spectrum, therefore, let us suppose that the impatient parties do not respond to investment by patient parties with any spending increases at all, and instead simply spend according to the impatient "warm-glow" schedule. In this case the patient do best to invest for 424 years—at which point they hold 46% of global wealth—and subsequently to implement the patient-optimal spending schedule. The patient payoff ratio is then 580.

In short, it appears, at least on this highly stylized analysis, that while the world remains predominantly comparatively impatient, patient philanthropists hoping to do good by increasing people's consumption might do up to several hundred times "more good" by investing the entirety of their resources on the current margin than by spending their resources on an impatient schedule.

Comparing the defection and warm-glow payoffs to the payoffs under common spending at the compromise rate α_C yields similarly large payoff ratios of 179 and

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[14]Gross world product was estimated to be $88T in 2019 by World Bank (2021a). Note that estimating global wealth in this way yields a substantially higher figure than a direct estimate of wealth holdings, e.g. that of Credit Suisse Research Institute (2021). This is primarily because the latter incorporates only explicit asset holdings, whereas the former also implicitly incorporates all human capital and untitled environmental resources.

[15]A thorough accounting of patient philanthropic assets would be beyond the scope of this paper, but there are at least two multibillionaires with close ties to the EA community who plan to give away the substantial majority of their wealth: Dustin Moskovitz and Sam Bankman-Fried, with estimated net worths of $17.8B and $8.7B respectively (Dolan et al., 2021). Even if they do not spend all their wealth philanthropically, or explicitly wish to allocate some of their philanthropic spending impatiently, the presence of many smaller EA philanthropists and donors renders it unlikely that the "patient philanthropic budget" is less than $25B.

356 respectively. Thus, by the reasoning of Corollary 2.2, a disbursement minimum of α_C reduces patient parties' payoffs to engaging in philanthropy here by more than two orders of magnitude.

In practice, philanthropists seeking to maximize global welfare do not face a choice between giving to a "representative agent" now or investing to do so in the future, but a choice between giving to the world's poorest today and investing to give in the future. That is, by taking B_I to be global wealth, rather than the wealth of the world's poorest, we are implicitly comparing a scenario in which the patient invest to fund consumption for everyone in the future to one in which they spread their wealth across the world today; but, whereas long-term investment at this scale can only rationally end in global and "untargeted" expenditures, by spending more quickly a philanthropist can target the world's poorest today. To roughly evaluate the magnitude of this consideration, however, observe that

$$\left(\frac{365}{9000}\right)^{-1.25} \approx 55 \ll 291 : \tag{38}$$

given $\eta = 1.25$, the marginal utility in consumption of an individual consuming \$1 per day is ("only") about 55 times greater than to that of an individual consuming the current average global consumption level of approximately \$9,000 per year (World Bank, 2021b). For a patient and impartial philanthropist, therefore, investing for the future appears to be worthwhile even when it comes at the cost of foregoing all targeting. The extent to which the world today neglects its poor is outweighed, not dramatically but substantially, by the extent to which it neglects its future.

An obstacle to taking disbursement plans on this scale literally is of course that, as the saving rate rises to $r - \alpha_P$ and absolute collective budget B rises to infinity, macroeconomic parameters here taken as exogenous should be expected to change. For instance, by assuming constant r, we implicitly assume an AK economy; in reality, given a multi-factor production function with diminishing returns to each factor, accelerated capital accumulation should lower interest rates (but offer the positive externality of raising other factor rents, such as wages). In this and other ways, further work would be necessary to embed the public good provision model developed here in a more realistic economic environment. Nevertheless, the exercise of this section demonstrates that patient philanthropic actors in an impatient world can indeed do well to invest most or all their funds for the relatively long run, and that spending impatiently—whether because they are legally required to or because they behaviorally set their disbursement rates to match those of their fellows—can constitute a highly significant error. If the details of the calibration as presented here fail, that is because the qualitative argument is a victim of its own success, recommending investment beyond the scale that a standard public good provision model is equipped to consider.

By contrast, let us now briefly consider a public good for which the patient and impatient budgets are initially equal. We will not alter the other parameters.

In this case, if the parties engage in the defection equilibrium, the patient invest for 30 years and subsequently implement the patient-optimal spending schedule, and the patient payoff ratio is a comparatively modest 1.36. If the impatient follow the warm-glow schedule, the patient can implement the patient-optimal spending schedule immediately, earning payoff ratio 1.44.

Note that these values depend only on the budget ratio and so are independent of the absolute budget sizes.

Lastly, of course, as $b_P \to 1$, the patient will begin spending immediately or almost immediately; the payoff ratios approach 1; and the spending schedule approaches that of §2.

6   Conclusion

The economic implications of time preference heterogeneity have been extensively explored in a variety of domains, including social discounting, optimal taxation, and dynamic bargaining. They have to date, however, largely been overlooked in literature on the private provision of public goods. The results presented here begin to fill this gap, and illustrate the importance of building time preference heterogeneity into models of dynamic public good provision going forward.

I have argued that in dynamic public good provision contexts, time preference heterogeneity is both even more widespread and even more important than in other contexts. Analyses of dynamic public good provision under common discounting assumptions therefore often risk being highly misleading.

For the most important dynamic public good provision games in practice, there is unusually clear empirical and theoretical evidence that their players do not employ common discounting. Models of dynamic public good provision are especially relevant in the context of international contributions to global public goods, because, in the absence of strong international governance, nations must effectively engage in public good provision games. While governments' time preference rates are typically chosen on the basis of domestic interest rate data, so as to match the time preferences of their own populations—eliminating time preference heterogeneity in games between governments and their constituents, at least if the constituents are themselves homogeneous—nations' time preferences often differ substantially from one another, as can be verified immediately from their respective published guidelines on discounting for public policy.

We should likewise expect time preference heterogeneity to be a pervasive feature of private philanthropy. Given the differing patience exhibited in households'

intertemporal consumption decisions, there is no reason to suppose that these differences would be absent among private parties seeking to fund a good that happens to be nonexcludable and nonrivalrous. Furthermore, philanthropy prototypically involves providing consumption goods to others who also to some extent provide for themselves, and as noted briefly in §5, time preference heterogeneity is almost intrinsic to the interaction between these two providers. This is because beneficiaries typically face mortality risks, and temptations to impatience, that we should not generally expect to appear (and which, at least to some extent, empirically do not appear) in the utility function of a third-party provider.

Again, moreover, time preference heterogeneity is not just a particularly widespread phenomenon in dynamic public good provision settings but also a particularly important one. As a comparison between the §2 and §3 models illustrates, equilibrium behavior in dynamic public good games with versus without time preference heterogeneity can differ dramatically: a player might spend at a strictly positive rate at time zero under homogeneity but, given a time preference rate only slightly below that of his co-funder, follow some period of spending nothing whatsoever. Behavior in equilibrium can then resemble behavior in a static game, at least under one particularly simple equilibrium—in which the parties' funding does not overlap across periods, so that each party only funds the goods with which he or she is relatively more concerned—but with a "bias" equivalent to giving impatient parties a first-mover advantage in the static setting. Nevertheless, even under arbitrarily small time preference differences, understanding and correctly implementing strategic behavior can in theory have arbitrarily large proportional welfare implications for small patient actors in a large impatient world, in the sense explored in §4 (though, interestingly, not the reverse). These implications may be large in practice as well, as computed roughly in §5.

These implications are more economically significant than the (still non-trivial) sensitivity of payoffs to spending rate choices when there is a single funder (as in the context of private goods), as can be seen by comparing the results of §2.2 and §4.3. Furthermore, from the perspective of a patient party, these implications do not rely on strong assumptions that the corresponding impatient party is perfectly rational and perfectly informed of the patient party's strategy. Indeed, they obtain even (and especially) if the impatient funder does not respond to his strategy at all, but simply spends as if she were the only funder, as found in §3.2 and §4.

The pervasiveness and importance of time preference heterogeneity in public good provision contexts has, in turn, at least two broad classes of policy implications.

First, it affects the structure of self-enforcing agreements for the provision of public goods among multiple parties who cannot contract, such as national governments. Such agreements can be efficient, as shown in §3.3, at least in continuous time and with perfect monitoring. If agreements are designed without accounting for the parties' different discount rates, however, they will generally not be efficient,

and may fail to be self-enforcing as intended. The United States's 2017 withdrawal from the Paris Agreement on climate change, for instance, coincided with an explicit impatient shift in Office of Management and Budget policies on social discounting, as requested by the newly elected president. If Americans are simply less patient than their international counterparts, this will in the long run affect democratically-set discounting policy, and punishments (here taking the form of increases in collective emissions) sufficient to deter defections by many countries may be insufficient to deter defections by Americans.

An understanding of the importance of time preference heterogeneity for public good provision might also affect the policy conversation around disbursement requirements for philanthropic foundations, trusts, and DAFs. Voters and policy-makers might look more charitably on slow- or non-disbursing charitable investment vehicles once a lack of disbursement is understood not as proof of a nefarious tax-avoidance scheme but as a natural consequence of patient philanthropic planning. Estimates of a patient philanthropist's willingness to pay to avoid disbursement requirements in principle—which, as shown in §4.3, can be arbitrarily higher in a multi-funder context under time preference heterogeneity than in the single-funder context (§2.2)—may also motivate patient philanthropists themselves to fight disbursement requirements more vigorously.

Of course, the results and discussions here only begin to cover the space of possible implications of time preference heterogeneity for public good provision.

The simple setting I have explored here focuses on the implications of time preference heterogeneity *per se* by featuring only two players and a single public good; perfect monitoring; no option of outside spending on private goods; a shock-free AK economy; and no way for spending to "do good" except through immediate "consumption". Even in this highly restricted setting, the analysis is incomplete without further work on equilibrium characterization and selection. To make precise predictions of what equilibrium will obtain in the absence of coordination, it would be necessary to determine whether the Defection Equilibrium or another SPE satisfies a natural equilibrium refinement. Likewise, to set a true lower bound on the payoffs that could accrue in equilibrium to each party, the equilibrium payoff set would have to be characterized more carefully. Finally, to quantify the benefits of coordination, it could be valuable to explore equilibrium selection among efficient SPEs, perhaps using tools from the considerable existing literature on dynamic bargaining under time preference heterogeneity.

Much more work, both theoretical and applied, is necessary to explore the implications of time preference heterogeneity in real-world public good provision problems which do not conform to the list of restrictions above. For instance, in reality, governments and philanthropists generally have more options than to fund present consumption goods and to invest for the funding of future consumption goods. They can also, say, fund present projects with high expected future payoffs, such as the

development of green technologies. Indeed, funders expressing patience often judge it worthwhile to fund such projects. Answering to what extent they should fund such projects immediately and to what extent they should invest for future funding—in light of other potential technology funders with different rates of time preference— would be a valuable application for future research.

Even the very simple model explored here confronts a host of complexities, in part because many useful results from the literature on repeated games (like algorithms to characterize the set of equilibrium payoffs) cannot be used here. Extensions along the lines above would doubtless face even more difficulties. Given the pervasive importance of time preference heterogeneity to dynamic public good provision, however, such efforts appear to be worthwhile.

References

Alan, Sule and Martin Browning, "Estimating Intertemporal Allocation Parameters using Synthetic Residual Estimation," *Review of Economic Studies*, 2010, *77* (4), 1231–1261.

Andreoni, James, "Impure Altruism and Donations to Public Goods: A Theory of Warm-Glow Giving," *The Economic Journal*, 1990, *100* (1), 464–477.

_ , "The Benefits and Costs of Donor Advised Funds," *Tax Policy and the Economy*, 2018, *32*, 1–44.

Arbesman, Samuel, "The Life-Spans of Empires," *Scholarly Incursions*, 2011, *44* (3), 127–129.

Barrage, Lint, "Be Careful What You Calibrate For: Social Discounting in General Equilibrium," *Journal of Public Economics*, 2018, *160*, 33–49.

Becker, Robert A., "On the Long-run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households," *Quarterly Journal of Economics*, 1980, *95* (2), 375–382.

Belfiori, Maria E., "Carbon Pricing, Carbon Sequestration, and Social Discounting," *European Economic Review*, 2017, *96*, 1–17.

Bergstrom, Theodore, Lawrence Blume, and Hal Varian, "On the Private Provision of Public Goods," *Journal of Public Economics*, 1986, *29* (1), 25–49.

Chambers, Christopher P. and Federico Echenique, "On Multiple Discount Rates," *Econometrica*, 2018, *86* (4), 1325–1346.

Credit Suisse Research Institute, "Global Wealth Report 2021," `https://www.credit-suisse.com/about-us/en/reports-research/global-wealth-report.html` (accessed 28 June 2021), 2021.

Dolan, Kerry A., Jennifer Wang, and Chase Peterson-Withorn, "World's Billionaires List: The Richest in 2021," `https://www.forbes.com/billionaires/` (accessed 27 June 2021), 2021. Forbes.

Drupp, Moritz, Mark Freeman, Ben Groom, and Frikk Nesje, "Discounting Disentangled," *American Economic Journal: Economic Policy*, 2018, *10* (4), 109–134.

Dutta, Prajit, "Dynamic Games with an Application to Climate Change Models," in Robert A. Myers, ed., *Encyclopedia of Complexity and Systems Science*, New York: Springer, 2017.

Farhi, Emmanuel and Ivan Werning, "Inequality and Social Discounting," *Journal of Political Economy*, 2007, *115* (3), 365–402.

__ **and** __ , "Progressive Estate Taxation," *Quarterly Journal of Economics*, 2010, *125* (2), 635–673.

Ferrari, Giorgio, Frank Riedel, and Jan-Henrik Steg, "Continuous-Time Public Good Contribution under Uncertainty," 2015. Working paper, Center for Mathematical Economics, Bielefeld University.

Fishman, Ram, "Heterogeneous Patience, Bargaining Power and Investment in Future Public Goods," *Environmental and Resource Economics*, 2019, *73*, 1101–1107.

Freeman, Mark, Ben Groom, and Michael Spackman, "Social Discount Rates for Cost-Benefit Analysis: A Report for HM Treasury," 2018. Summary report, London School of Economics.

Frietas-Groff, Zachary, "Climate Warriors and Deficit Hawks: Eliciting Intergenerational Discount Rates," 2020. Unpublished manuscript.

Gollier, Christian and Richard Zeckhauser, "Aggregation of Heterogeneous Time Preferences," *Journal of Political Economy*, 2005, *113* (4), 878–96.

Heal, Geoffrey M. and Antony Millner, "Agreeing to Disagree on Climate Policy," *Proceedings of the National Academy of Sciences*, 2014, *111* (10), 3695–3698.

HM Treasury, "The Green Book: Central Government Guidance on Appraisal and Evaluation," `https://assets.publishing.service.gov.uk/government/uploads/system/uploads/attachment_data/file/938046/The_Green_Book_2020.pdf` (accessed 10 Feb 2020), 2020.

Jackson, Matthew O. and Leeat Yariv, "Collective Dynamic Choice: The Necessity of Time Inconsistency," *American Economic Journal: Microeconomics*, 2015, *7* (4), 150–178.

Jacobsen, Mark, Jacob LaRiviere, and Michael Price, "Public Policy and the Private Provision of Public Goods under Heterogeneous Preferences," *Journal of the Association of Environmental and Resource Economists*, 2017, *4* (1), 243–280.

Kalai, Adam and Ehud Kalai, "Strategic Polarization," *Journal of Mathematical Psychology*, 2001, *45*, 656–663.

Koopmans, Tjalling C., "Objectives, Constraints, and Outcomes in Optimal Growth Models," *Econometrica*, 1967, *35* (1), 1–15.

Laibson, David, "Self-control and Saving," 1994. Technical report, Harvard University.

Millner, Antony, "Nondogmatic Social Discounting," *American Economic Review*, 2020, *110* (3), 760–775.

__ **and Geoffrey Heal**, "Discounting by Committee," *Journal of Public Economics*, 2018, *167*, 91–104.

National Philanthropic Trust, "The 2020 DAF Report," `https://www.nptrust.org/reports/daf-report/` (accessed 30 June 2020), 2020.

Pharoah, Catherina and Jenny Harrow, "Payout with an English accent: Exploring the case for a foundation 'distribution quota' in the UK," 2010. Working Paper, ARNOVA 39th Annual Conference (Alexandria, USA).

Rader, Trout, "Utility over Time: The Homothetic Case," *Journal of Economic Theory*, 1981, *25* (2), 219–236.

Ramsey, Frank P., "A Mathematical Theory of Saving," *The Economic Journal*, 1928, *38* (152), 543–559.

Reich, Robert, *Just Giving*, Princeton: Princeton University Press, 2018.

Ryder, Harl, "Heterogeneous Time Preferences and the Distribution of Wealth," *Mathematical Social Sciences*, 1985, *9* (1), 63–76.

Sandberg, Anders, "Everything Is Transitory, for Sufficiently Large Values of 'Transitory'," 2019. Lecture, 3rd Oxford Workshop on Global Priorities Research (Oxford, UK).

Simon, Leo K. and Maxwell B. Stinchcombe, "Extensive Form Games in Continuous Time: Pure Strategies," *Econometrica*, 1989, *57* (5), 1171–1214.

Sorger, Gerhard, "Recursive Nash Bargaining over a Productive Asset," *Journal of Economic Dynamics and Control*, 2006, *30* (12), 2637–2659.

Stern, Nicholas, *The Economics of Climate Change*, New York: Cambridge University Press, 2006.

Stinchcombe, Maxwell B., "Continuous Time Game Theory: An Infinitesimal Approach," `http://www.laits.utexas.edu/~mbs31415/InfGTc.pdf` (accessed 23 Sep 2020), 2013.

US Office of Management and Budget, "Circular A-94: Guidelines and Discount Rates for Benefit-Cost Analysis Of Federal Programs," `https://www.whitehouse.gov/sites/whitehouse.gov/files/omb/circulars/A94/a094.pdf` (accessed 10 Feb 2020), 2019.

von Below, Nicolaus, "Optimal Carbon Taxes with Social and Private Discounting," 2012. Working Paper, SURED Conference (Ascona, Switzerland).

Weitzman, Martin L., "Why the Far-Distant Future Should Be Discounted at Its Lowest Possible Rate," *Journal of Environmental Economics and Management*, 1998, *36* (3), 201–208.

World Bank, "GDP (current USD): World," `https://data.worldbank.org/indicator/NY.GDP.MKTP.CD` (accessed 27 June 2021), 2021.

___ , "Households and NPISHs Final consumption expenditure per capita (constant 2010 USD): World," https://data.worldbank.org/indicator/NE.CON.PRVT.PC.KD (accessed 29 June 2021), 2021.

A Framework for games in continuous time

A.1 Preliminaries

Throughout this paper we consider dynamic games and optimization problems continuous time. We roughly follow the framework introduced by Simon and Stinchcombe (1989), as also exposited by Stinchcombe (2013) and as summarized here.

Note that Simon and Stinchcombe (1989) simplify their (otherwise much more general) analysis by permitting players only finitely many changes in the actions they play, and Stinchcombe (2013) simplifies his exposition of the relevant material by restricting the time horizon to $[0, 1)$ (without explicitly noting, as Simon and Stinchcombe (1989) do, that an analogous framework can also be used over an infinite horizon). The relevant results, however, also apply in our case when players are permitted countably many action changes and face an infinite horizon. Later work relaxes both these restrictions, but only within the context of repeated games.

Partition the nonnegative half of the real line by a sequence of grids, indexed by n, such that grid $n+1$ is a strict refinement of grid n. The elements of grid n are of equal length, and this length converges uniformly to zero as n increases. For our purposes, the choice of grid sequence will not be significant, within the caveats noted below. But where not otherwise specified, we will use the grid sequence G characterized by

$$G_n \triangleq \left\{ \left[kg_n, (k+1)g_n \right) \right\}_{k=0}^{\infty}, \text{ where } g_n = 10^{-n}. \tag{39}$$

The kth element of G_n will be denoted $G_{n,k}$. The set $\{kg_n\}_{k=0}^{\infty}$ will be denoted $\underline{G}_n$, likewise indexed by k. As we can see, the length of G's elements at n takes a value, denoted g_n (here equaling 10^{-n}), which tends to zero as n increases. The index of the element of G_n containing time t will be denoted $k_n(t)$. We will denote the set $\cup_{n=0}^{\infty} \underline{G}_n$ by $\underline{G}_{\infty}$, and we will call $\underline{G}_{\infty}$ G's "grid points".

We will define a dynamic game or optimization problem in continuous time to be the limit, in a certain sense, as $n \to \infty$ of the sequence of corresponding games or problems in discrete time, where game or problem n is played on grid G_n.

A.2 Dynamic optimization problems

All the continuous-time dynamic optimization problems we consider will be consumption allocation problems. More precisely, they will take the form

$$\max_{\{Y(t) \geq 0\}_{t \in [0,\infty)}} U(Y) \;\; : \;\; \int_0^{\infty} Y(t) \leq B, \;\; U(Y) = \int_0^{\infty} e^{-\delta t} u(Y(t), t) dt \tag{40}$$

for some (perhaps time-varying) flow utility function u, time preference rate δ, and interest rate r, where B is the agent's budget at time zero. $Y(t)$ denotes the rate at which resources are allocated for investment until, followed by spending at, t.

For a given problem (u, δ, B) and a given grid sequence G, corresponding discrete-time problem n is identical except in that it imposes the restriction that the allocation rate $Y(t)$ be constant throughout each member of G_n. More precisely, corresponding discrete-time problem n takes the form

$$\max_{\{Y(kg_n)\}_{k=0}^{\infty}} \sum_{k=0}^{\infty} \int_{kg_n}^{(k+1)g_n} e^{-\delta t} u(Y(kg_n), t) dt \tag{41}$$

$$: \sum_{k=0}^{\infty} \int_{kg_n}^{(k+1)g_n} Y(kg_n) dt \leq B.$$

We will say that the solution to a dynamic optimization problem in continuous time is spending schedule $Y^*(t)$ if the optimal spending rate at t in corresponding problem n converges to $Y^*(t)$ as $n \to \infty$.

For our purposes, solutions to dynamic optimization problems in continuous time as defined thus are equal in every respect to those as defined using the standard tools of control theory, with two exceptions.

First, by defining the optimal spending rate at t in a continuous-time optimization problem as above, we render it undefined when the limit of optimal spending rates at t across the sequence of problems generated by the given grid does not exist. This can happen only when an optimal spending schedule in the continuous-time problem, as conventionally defined, has a discontinuity at t.

For example, consider the optimization problem defined above but with

$$u(X(t), t) = \begin{cases} \sqrt{(X(t))}, & t \leq 0.\overline{90}; \\ 0, & t > 0.\overline{90}. \end{cases} \tag{42}$$

$0.\overline{90}$ is a discontinuity point of this optimization problem. Using grid G as defined in (39), the element containing $0.\overline{90}$ in grid n is

$$\left[\sum_{k=1}^{n/2} 9 \cdot 10^{1-2k}, \sum_{k=1}^{n/2} 9 \cdot 10^{1-2k} + 10^{-n} \right), \qquad n \text{ even}; \tag{43}$$

$$\left[\sum_{k=1}^{(n+1)/2} 9 \cdot 10^{1-2k}, \sum_{k=1}^{(n-1)/2} 9 \cdot 10^{1-2k} + 10^{1-n} \right), \qquad n \text{ odd}.$$

The optimal spending rate at $t^* = 0.\overline{90}$ in discrete-time problem n converges to a value slightly below 1 across the problem subsequence indexed by even n (in which $0.\overline{90}$ lies near the top of the interval), and to a value slightly above 0 across the problem subsequence indexed by odd n (in which $0.\overline{90}$ lies near the bottom of the

interval). Thus, using G, the optimal spending rate at $t = 0.\overline{90}$ in the continuous-time problem is undefined.

If possible, this issue can be resolved by choosing a grid sequence $\tilde{G}$ such that

$$t^* \in \underline{\tilde{G}}_\infty \tag{44}$$

for all discontinuity points t^*. In the problem above, for example, we may use the grid sequence characterized by $\tilde{g}_n = 0.\overline{90} \cdot 10^{-n}$. Because $G_n(0.\overline{90})$ is now comprised almost entirely of points strictly greater than $0.\overline{90}$, for all $n \geq \underline{n} = 0$, we can define $X^*(0.\overline{90}) = 0$.

The continuous-time optimization problems considered throughout this paper have at most one discontinuity point t^*. We will always implicitly define such a problem by a grid sequence satisfying (44) with respect to t^*, so that the optimal allocation is defined to be at least right-continuous; i.e., so that the optimal allocation rate at a discontinuity point t^*, if one exists, is defined to be equal to the limit of optimal spending rates at $t^* + \epsilon$ as $\epsilon \downarrow 0$.

Second, we can introduce a term "dt" which functions in certain natural ways as infinitesimal time unit: one that is held to be strictly positive but strictly less than any positive real number. Strictly speaking, dt will denote g: the sequence of grid element lengths, tending to zero, induced by the given grid sequence G. Given G as defined in (39), for instance, $dt = \{1, \frac{1}{10}, \frac{1}{100}, ...\}$. It thus denotes the lengths of time between spending rate changes that an agent faces across the sequence of dynamic optimization problems generated by G.

This convention allows us to intuitively determine that infinitesimal deviations from an agent's optimal spending schedule which cannot be obtained as the limit of optimal spending schedules over the discrete grid will not obtain in equilibrium, because they offer the agent a strictly (albeit "infinitesimally") lower payoff or are strictly ("infinitesimally") infeasible. For example, consider an agent who can allocate a budget of 1 over time, without interest or time preference, and whose flow utility in the allocation at t, denoted $u(Y(t), t)$, is given by

$$u(Y(t), t) = \begin{cases} \sqrt{Y(t)}, & t < 1; \\ 0, & t \geq 1. \end{cases} \tag{45}$$

That is, in effect, she must choose $Y(t)$ for $t \in [0, 1)$ to maximize

$$\int_0^1 \sqrt{Y(t)} dt \quad : \quad \int_0^1 Y(t) dt \leq 1. \tag{46}$$

It is clear that she maximizes her utility by choosing

$$Y^*(t) = \begin{cases} 1, & t < 1; \\ 0, & t \geq 1. \end{cases} \tag{47}$$

She also maximizes her utility, however, by deviating from $Y^*(t)$ on a measure-zero set of times—e.g. by choosing

$$Y(t) = \begin{cases} 1, & t \in [0,1) \backslash \{\frac{1}{2}\}; \\ 0, & t = \{\frac{1}{2}\} \cup [1, \infty). \end{cases} \tag{48}$$

Whereas Y^* can be obtained as the limit of optimal allocations across discrete-time problems n, however, Y cannot. Instead of explicitly proving that a spending rate of 0 does not obtain at $t = \frac{1}{2}$ for a given sequence of discrete-time games, we can simply observe that schedule Y offers the agent dt less utility than Y^*. That is, for each discrete-time problem n in the sequence, there is a positive utility loss of g_n (the width of the interval containing $\frac{1}{2}$) times $\sqrt{1} = 1$ (the foregone flow utility throughout that interval).

Likewise, we might consider the alternative allocation

$$Y(t) = \begin{cases} 1, & t \in [0,1) \backslash \{\frac{1}{2}\}; \\ 4, & t = \frac{1}{2}; \\ 0, & t \in [1, \infty). \end{cases} \tag{49}$$

While this allocation would offer dt higher utility, across the sequence of discrete-time problems, it is infeasible: it costs $1 + 3dt > B$.

Choose a grid sequence G satisfying (44) with respect to any discontinuity point t^*. Despite the expository examples above, throughout this paper we will consider only problems in which flow utility $u(Y(t), t)$ is twice differentiable, strictly increasing, and strictly concave in its first argument. The optimal allocation Y^* is then characterized by the condition that, for some $\lambda > 0$,

$$\frac{\partial U}{\partial Y(t)}(Y^*) \leq \lambda, \ \ Y^*(t) = 0; \tag{50}$$

$$\frac{\partial U}{\partial Y(t)}(Y^*) = \lambda, \ \ Y^*(t) > 0$$

for all $t \in \underline{G}_\infty$, where

$$\frac{\partial U}{\partial Y(t)}(Y^*) = \begin{cases} e^{-\delta t} \frac{\partial U}{\partial Y(t)}(Y^*(t), t) \, dt, & t \neq t^*; \\ \\ \lim_{\epsilon \downarrow 0} e^{-\delta t} \frac{\partial U}{\partial Y(t)}(Y^*(t), t + \epsilon) \, dt, & t = t^*, \end{cases} \tag{51}$$

where t^* denotes a discontinuity point, if one exists. By the right-continuity of $Y^*(t)$, the above conditions identify $Y^*(t)$ across all t, not just grid points.

On verifying the above conditions, we will also be verifying the undesirably or infeasiblity of all "infinitesimal" deviations from Y^*. We can characterize Y^* as the *unique* solution to the continuous-time problem, rather than merely a solution unique up to measure-zero deviations.

A.3   Dynamic games

The dynamic game we consider in §3.3 takes the following form. There is a grid sequence G, as described in A.2; two players $i = I, P$; and an action set $\mathbb{X}_i$ for each player governing the actions i takes—in particular, the rates at which i spends—at a given time. A history is an assignment of an action $X_{i,t}$ to each player i for each $t \geq 0$. It will be denoted by

$$X \triangleq \{(X_{I,t}, X_{P,t})\}_{t=0}^{\infty}. \tag{52}$$

In a slight abuse of notation, partial histories of a given history X, defined only on subsets of the real line, will be denoted by set-subscripts on X. An open history from $t = 1$ to 2, for instance, will be denoted by $X_{(1,2)}$. For simplicity, a partial history truncated just before some t may be denoted by $X_{|t}$ (rather than $X_{[0,t)}$). X_t will denote the *total* spending rate at t; the partial history defined only at t, i.e. the action profile at t, would be denoted by $X_{\{t\}}$. Partial histories will be conjoined with commas; so, for example, $(X_{|t}, \tilde{X}_{[t,t+1)})$ denotes the assignment of actions given by X at times before t and by an alternative history $\tilde{X}$ from t to just before $t + 1$.

Each player i has a budget at $t = 0$ of B_i and a history-dependent budget at t of $B_i(X_{|t})$.

Grid sequence G, with its sequence of grid element lengths g, generates a sequence of discrete-time games, also indexed by n. In game n, spending rates $X_{i,t}$ are chosen simultaneously by each player i at each $t \in \underline{G}_n$, and must be maintained by the respective players throughout $G_{n,k_n(t)}$.

Open partial history $X_{|t}$ is "feasible" if $B_i(X_{|t}) \geq 0 \; \forall i$. The set of decision nodes in game n is the set of open partial feasible histories ending just before a time $t \in \underline{G}_n$. We will denote this set by $\mathbb{D}_n$.

Spending rate A is "feasible" for i at node $X_{|t}$ if $Ag_n \leq B_i(X_{|t})$. We will denote the set of feasible actions at $X_{|t} \in \mathbb{D}_n$ by $\mathbb{X}_i(X_{|t}, n)$.

A strategy σ_i for player i in game n is a function $X_{|t} \mapsto \mathbb{X}_i(X_{|t}, n)$ (for $X_{|t} \in \mathbb{D}_n$) from nodes to feasible actions. The set of i's strategies will be denoted by Σ_i, and the set of all strategy profiles by Σ.

From node $X_{|t}$, subsequent adoption of strategy profile σ generates a unique and feasible history, which we will denote $\chi(X_{|t}, \sigma)$. This can be shown recursively: $\chi(X_{|t}, \sigma)_{G_{n,k_n(t)}}$ is feasible and uniquely determined by $X_{|t}$, and $\chi(X_{|t}, \sigma)_{G_{n,k_n(t)+m}}$ is likewise feasible and uniquely determined by $\chi(X_{|t}, \sigma)_{[0, \underline{G}_{n,k_n(t)+m})}$ for all $m \geq 1$.

As an important special case of this observation, note that a complete feasible history is uniquely determined by any strategy profile σ. We may denote this history $\chi(\sigma)$, omitting the first argument (which would be $X_\emptyset$) for simplicity.

Utility for player i as a function of the history, denoted $U_i(X)$, is a δ_i-discounted function of i's flow utility u_i. Flow utility for i at t is a function of the total spending

rate at t:

$$U_i(X) = \int_0^\infty e^{-\delta_i t} u_i(X_t) dt. \tag{53}$$

A subgame-perfect equilibrium of game n is a strategy profile σ^* such that, for all $X_{|t} \in \mathbb{D}_n$,

$$\int_t^\infty e^{-\delta_i(s-t)} u(\chi(X_{|t}, \sigma^*)_s) ds \tag{54}$$

$$\geq \int_t^\infty e^{-\delta_i(s-t)} u(\chi(X_{|t}, (\sigma_i, \sigma_j^*))_s) ds \quad \forall \sigma_i \in \Sigma_i$$

for both players i (where j denotes the other player).

In this context, the move to continuous time, given grid sequence G, is relatively straightforward.

The set of decision nodes in the continuous-time game is the set of open partial feasible histories ending just before a time $t \in \underline{G}_\infty$, denoted $\mathbb{D}_\infty$. A strategy for i in the continuous-time game is a function $\sigma_i : \mathbb{D}_\infty \to \mathbb{R}$ assigning spending rates to nodes such that, for some sequence of strategies $\{\sigma_i^n\}$ across discrete-time games n, $\sigma_i^n(X_{|t}) \to \sigma_i(X_{|t}) \ \forall X_{|t} \in \mathbb{D}_\infty$.

A subgame-perfect equilibrium σ^* of the continuous-time game is a strategy profile (σ_I^*, σ_P^*) of the continuous-time game approached (pointwise) by a sequence of discrete-time strategy profiles $\{\sigma^n\}$ such that σ^n is a subgame-perfect equilibrium of game n for all $n \geq 0$.

A framework along these lines is necessary when defining dynamic games in continuous time in order for strategy profiles σ to generate histories $X(\sigma)$, defined at least on $\underline{G}_\infty$. If strategies in continuous time were defined as arbitrary functions from open partial feasible histories to spending rates (subject only to some feasibility constraint), the history generated by a given strategy profile could be almost completely indeterminate. Consider for instance the strategy

$$\sigma_i(X_{|t}) = \begin{cases} 0, & X_{I,s} = 0 \ \forall s < t; \\ 1, & \exists s < t : X_{I,s} \neq 0 \end{cases} \tag{55}$$

(assuming that this is feasible). In addition to the obvious possibility that $X_{I,s} = 0 \ \forall s$, any history X such that $X_{I,s} = 0 \ \forall s \leq t$ and $X_{I,s} = 1 \ \forall s > t$, for some $t \geq 0$, is compatible with σ_i.

Even under the framework outlined above, however, spending behavior at times $t \notin \underline{G}_\infty$ may not be defined by a given strategy profile σ—even if σ is a subgame-perfect equilibrium, as defined above. Consider for example the utility function

$$u(X_t) = \begin{cases} 1, & X_t = 1; \\ 0, & X_t \neq 1. \end{cases} \tag{56}$$

The following discrete-time strategy profile produces a well-defined history, and is a subgame-perfect equilibrium, for all games n:

$$\sigma_I^n(X_{|t}) = \begin{cases} 1, & 0.\overline{90} > \underline{G}_{n,k_n(t)} + \frac{g_n}{2}; \\ 0, & \text{otherwise}, \end{cases} \tag{57}$$

$$\sigma_P^n(X_{|t}) = \begin{cases} 0, & 0.\overline{90} > \underline{G}_{n,k_n(t)} + \frac{g_n}{2}; \\ 1, & \text{otherwise}. \end{cases} \tag{58}$$

Using G as defined in (39), however, by reasoning analogous to that in the similar example in A.2, each player's spending at $t = 0.\overline{90}$ oscillates between 0 and 1, depending on the parity of n. Though the limiting strategy profile σ^* is well-defined, therefore, $\chi(\sigma^*)_{0.\overline{90}}$ is not.

More generally, given a continuous-time strategy profile σ, $\chi(\sigma)_t$ may be undefined when $\chi(\sigma)$ (as defined by some sequences σ^n and G) is discontinuous at t. As in A.2, let us call such times "discontinuity points". The dynamic game subgame-perfect equilibria σ^* we consider will have at most one discontinuity point t^*. By choosing G such that (44) holds with respect to t^*, we can ensure that $\chi(\sigma^*)$ is everywhere defined.

Having chosen such a G, and using the dt notation introduced in A.2, strategy profile σ^* is a subgame-perfect equilibrium in continuous time if, at all $X_{|t} \in \mathbb{D}_\infty$,

$$\int_t^\infty e^{-\delta_i s} u(\chi(X_{|t}, \sigma^*)_s) ds \geq \int_t^\infty e^{-\delta_i s} u(\chi(X_{|t}, (\sigma_i, \sigma_j^*))_s) ds \tag{59}$$

for both players i (where j denotes the other player), for all strategies σ_i such that $\chi(X_{|t}, (\sigma_i, \sigma_j^*))_s$ is defined for all $s > t$.

We can also characterize subgame-perfect equilibria in continuous time using an intuitive "one-shot deviation principle", so long as the game satisfies the standard condition of continuity at infinity: then strategy profile σ^* is a subgame-perfect equilibrium if, at all $X_{|t} \in \mathbb{D}_\infty$,

$$\int_t^\infty e^{-\delta_i s} u(\chi(X_{|t}, \sigma^*)_s) ds \tag{60}$$

$$\geq u(A + \sigma_j^*(X_{|t})) dt + \int_{t+dt}^\infty e^{-\delta_i s} u(\chi((X_{|t}, (A, \sigma_j^*(X_{|t}))_{q \in [t, t+dt)}), \sigma^*)_s) ds$$

for both players i (where j denotes the other player), for all spending rates A such that $\chi((X_{|t}, (A, \sigma_j^*(X_{|t}))_{[t, t+dt)}), \sigma^*)_s$ is defined for all $s > t$.

B Proofs

B.1 Proof of Proposition 1

Let

$$Y(t) \triangleq e^{-rt}X(t) \tag{61}$$

denote the resources allocated at time 0 for investment until, followed by spending at, t. Let

$$v_t(Y(t)) \triangleq e^{-\delta t}u(e^{rt}Y(t)) \tag{62}$$

denote the discounted flow utility at t from allocation $Y(t)$.

Since utility in spending is time-additive, differentiable, and strictly concave, allocation Y maximizes utility iff the marginal utility of allocating resoures to each $t \in \underline{G}_\infty$ equals λdt (see Appendix A.2) for some constant λ:

$$v'(Y(t)) = \frac{\partial}{\partial[Y(t)]}\left[e^{-\delta t}\frac{(e^{rt}Y(t))^{1-\eta}-1}{1-\eta}\right] = \lambda \; \forall t, \; \eta \neq 1;$$

$$= \frac{\partial}{\partial[Y(t)]}\left[e^{-\delta t}\ln(e^{rt}Y(t))\right] = \lambda \; \forall t, \; \eta = 1.$$

Taking the derivative and rearranging (and recalling the impossibility of discontinuities at $t \notin \underline{G}_\infty$), we have

$$Y^*(t) = \lambda^{\frac{-1}{\eta}}e^{\frac{r-r\eta-\delta}{\eta}t} \;\; \forall t \geq 0. \tag{63}$$

Subjecting this resource allocation to the budget constraint, we have

$$\int_0^\infty \lambda^{\frac{-1}{\eta}}e^{\frac{r-r\eta-\delta}{\eta}t}dt = B. \tag{64}$$

If $\delta > r(1-\eta)$, we find

$$\lambda = \left(B\frac{r\eta-r+\delta}{\eta}\right)^{-\eta}. \tag{65}$$

Then, substituting (65) into (63), and recalling that $X^*(t) = e^{rt}Y^*(t)$, we have

$$X^*(t) = B\frac{r\eta-r+\delta}{\eta}e^{\frac{r-\delta}{\eta}t}. \tag{66}$$

This is the optimal spending schedule.

The payoff to following this spending schedule is then

$$U = \begin{cases} \int_0^\infty e^{-\delta t}\dfrac{\left(B\frac{r\eta-r+\delta}{\eta}e^{\frac{r-\delta}{\eta}t}-1\right)^{1-\eta}}{1-\eta}dt, & \eta \neq 1; \\[4ex] \int_0^\infty e^{-\delta t}\ln\left(B\frac{r\eta-r+\delta}{\eta}e^{\frac{r-\delta}{\eta}t}\right)dt, & \eta = 1, \end{cases} \tag{67}$$

which simplify to

$$U = \begin{cases} \frac{B^{1-\eta}}{1-\eta}\left(\frac{r\eta-r+\delta}{\eta}\right)^{-\eta} - \frac{1}{\delta(1-\eta)}, & \eta \neq 1; \\\\ \frac{\delta\ln(B\delta)+r-\delta}{\delta^2}, & \eta = 1. \end{cases} \tag{68}$$

If $\delta \leq r(1-\eta)$ and $\eta > 0$, integral (64) is not defined for any λ. There is thus no optimal spending schedule.

B.2 Proof of Proposition 4

As in the proof of Proposition 1 (Appendix B.1), let

$$Y_P(t) \triangleq e^{-rt} X_P(t) \tag{69}$$

denote the resources the patient party allocates at time 0 for investment until, followed by spending at, t. Let $Y_I(t)$ be defined likewise, and let $Y(t) \triangleq Y_P(t) + Y_I(t)$.

Consider first the case in which I is warm-glow. Given that the impatient party follows allocation Y_I, let

$$v_{P,t}(Y(t)) \triangleq e^{-\delta_P t} u(e^{rt} Y(t)) \tag{70}$$

denote the patient party's discounted flow utility at t from allocation $Y_P(t)$.

From Proposition 1, and by the stipulation that I is warm-glow, I's optimal spending schedule is

$$Y_I(t) = B_I \alpha_I e^{(r-\alpha_I)t}, \tag{71}$$

independently of Y_P. Furthermore, the patient party's discounted flow utility in the collective allocation $Y(t)$ is time-additive, differentiable, strictly increasing, and strictly concave at each time t. Taking $Y_I(t)$ as given, therefore, the patient party maximizes his utility by setting $Y_P(t)$ such that he is indifferent to marginal resource reallocation across times to which he is allocating resources at a positive rate, and weakly prefers marginal resource allocation to these times to marginal resource allocation to other times. That is, differentiating (70),

$$\lambda_t(Y_I(t), Y_P(t)) \triangleq \frac{\partial}{\partial[Y_P(t)]} \Big[v_{P,t}(Y_I(t) + Y_P(t)) \Big] \tag{72}$$

$$= e^{-\alpha_P \eta t} (Y_P(t) + Y_I(t))^{-\eta}$$

$$= \lambda^* > 0 \qquad\qquad \text{if } Y_P(t) > 0;$$

$$\leq \lambda^* \qquad\qquad \text{if } Y_P(t) = 0.$$

Substituting (71) into (72), we have that if $Y_P(t) = 0$,

$$\lambda_t = \big(B_I \alpha_I \big)^{-\eta} e^{(\delta_I - \delta_P)t}. \tag{73}$$

As we can see, if $Y_P(t) = 0$, λ_t is strictly increasing in t. It follows from (72) that, if $Y_P(t) > 0$ for some t, $Y_P(s) > 0 \ \forall s > t$. That is, there is some t^* such that $y_P(t) = 0 \ \forall t < t^*$ and $Y_P(t) = 0 \ \forall t > t^*$.

Thus $v'_{P,t}(Y(t)) = \lambda^*$ is constant for all $t > t^*$. This implies that the collective allocation $Y(t)$ $(t > t^*)$ constitutes the patient-optimal allocation of the collective budget allocated to $t > t^*$.

This leaves us with two cases.

If $t^* = 0$, then $\lambda_t = \lambda^*$ $\forall t$, so Y constitutes the patient-optimal allocation of the collective budget. The impatient allocation rate of B_I at $t = 0$ must therefore not be greater than the patient allocation rate of the collective budget at $t = 0$. That is,

$$Y_I(0) = B_I \alpha_I \leq B \alpha_P. \tag{74}$$

If $t^* > 0$, note first that $Y_P(t)$ must be continuous. If there were some $\tilde{t}$ at which $Y_P(t)$ were discontinuous, then, since $Y_I(t)$ is continuous, $Y(t) = Y_P(t) + Y_I(t)$ would also be discontinuous at $\tilde{t}$. Because $v_{P,t}(Y(t))$ is continuous in t and $Y(t)$, it too would then be discontinuous at $\tilde{t}$. The patient party would then be able to increase his utility by reallocating marginal funds from $\tilde{t}$ to $\tilde{t} - dt$ or $\tilde{t} + dt$.

In particular, Y_P is continuous at t^*. Since $Y_P(t) = 0$ $\forall t < t^*$, it follows that $Y_P(t^*) = 0$.

Furthermore, since $\lambda_t(Y_P(t), Y_I(t))$ is continuous in $Y_P(t)$, $Y_I(t)$, and t, and since $\lambda_t(Y_P(t), Y_I(t)) = \lambda^*$ $\forall t > t^*$, we now have $\lambda_{t^*} = \lambda^*$. Thus $Y(t^*) = Y_I(t^*)$ constitutes the patient-optimal allocation rate of the collective resources remaining at t^*.

That is,

$$B_I \alpha_I e^{-\alpha_I t^*} = \left(B_P + B_I e^{-\alpha_I t^*}\right) \alpha_P. \tag{75}$$

Rearranging, we have

$$t^* = \ln\left(\frac{B_I}{B_P} \frac{\alpha_I - \alpha_P}{\alpha_P}\right)\bigg/ \alpha_I. \tag{76}$$

Now, multiplying both sides of (75) by $e^{\alpha_I t^*}$ and substituting (76) for t^*, we have

$$B_I \alpha_I = \left(B_P\left(\frac{B_I}{B_P} \frac{\alpha_I - \alpha_P}{\alpha_P}\right) + B_I\right)\alpha_P$$

$$> B \alpha_P, \tag{77}$$

because, from (76),

$$t^* > 0 \implies \frac{B_I}{B_P} \frac{\alpha_I - \alpha_P}{\alpha_P} > 1. \tag{78}$$

The inequality on $Y_I(0)$ provided by (74) and (77) thus characterizes whether $t^* = 0$ or $t^* > 0$. In particular, solving it for t^*, we have

$$t^* = \begin{cases} 0 & \frac{B_I}{B_P} \frac{\alpha_I - \alpha_P}{\alpha_P} \leq 1 \\ \ln\left(\frac{B_I}{B_P} \frac{\alpha_I - \alpha_P}{\alpha_P}\right)\bigg/ \alpha_I & \frac{B_I}{B_P} \frac{\alpha_I - \alpha_P}{\alpha_P} > 1 \end{cases} \tag{79}$$

which reduces to

$$t^* = \max\left(0, \ln\left(\frac{B_I}{B_P} \frac{\alpha_I - \alpha_P}{\alpha_P}\right)\bigg/ \alpha_I\right). \tag{80}$$

Finally, observe that the collective budget at t^* is

$$B_I e^{(r-\alpha_I)t^*} + B_P e^{rt^*} \tag{81}$$

in either case, and recall that $X_P(t)$ will fill the gap between $X_I(t)$ and the patient-optimal spending rate of the collective budget following t^*. It follows immediately that

$$X_P(t) = \begin{cases} 0, & t < t^*; \\ \left(B_I e^{(r-\alpha_I)t^*} + B_P e^{rt^*} \right) \alpha_P e^{(r-\alpha_P)(t-t^*)} - B_I \alpha_I e^{(r-\alpha_I)t}, & t \geq t^*. \end{cases}$$

Now consider the case in which P is warm-glow. Given that the patient party follows allocation Y_P, let

$$v_{I,t}(Y(t)) \triangleq e^{-\delta_I t} u(e^{rt} Y(t)) \tag{82}$$

denote the impatient party's discounted flow utility at t from allocation $Y_I(t)$.

From Proposition 1, and by the stipulation that P is warm-glow, P's optimal spending schedule is

$$Y_P(t) = B_P \alpha_P e^{(r-\alpha_P)t}, \tag{83}$$

independently of Y_I. Furthermore, the impatient party's discounted flow utility in the collective allocation $Y(t)$ is time-additive, differentiable, strictly increasing, and strictly concave at each time t. Taking $Y_P(t)$ as given, therefore, the impatient party maximizes her utility by setting $Y_I(t)$ such that she is indifferent to marginal resource reallocation across times to which she is allocating resources at a positive rate, and weakly prefers marginal resource allocation to these times to marginal resource allocation to other times. That is, differentiating (82),

$$\lambda_t(Y_I(t), Y_P(t)) \triangleq \frac{\partial}{\partial [Y_I(t)]} \left[v_{I,t}(Y_I(t) + Y_P(t)) \right] \tag{84}$$

$$= e^{-\alpha_I \eta t}(Y_I(t) + Y_P(t))^{-\eta}$$

$$= \lambda^* > 0 \qquad\qquad \text{if } Y_I(t) > 0;$$

$$\leq \lambda^* \qquad\qquad \text{if } Y_I(t) = 0.$$

Substituting (83) into (84), we have that if $Y_I(t) = 0$,

$$\lambda_t = \left(B_P \alpha_P \right)^{-\eta} e^{(\delta_P - \delta_I)t}. \tag{85}$$

As we can see, if $Y_I(t) = 0$, λ_t is strictly increasing in t. It follows from (84) that, if $Y_I(t) > 0$ for some t, $Y_I(s) > 0 \ \forall s > t$. That is, there is some t^* such that $Y_I(t) > 0 \ \forall t < t^*$ and $Y_I(t) = 0 \ \forall t > t^*$.

Thus $v'_{I,t}(Y(t)) = \lambda^*$ is constant for all $t < t^*$. This implies that the collective allocation $Y(t)$ $(t < t^*)$ constitutes the impatient-optimal allocation of the collective budget allocated to $t < t^*$; for some L_I,

$$\int_0^{t^*} L_I e^{-\alpha_I t} dt = B_I + B_P(1 - e^{-\alpha_P t^*}). \tag{86}$$

Furthermore, by arguments analogous to those following (74), $Y_I(t^*) = 0$. Thus t^* satisfies

$$L_I e^{-\alpha_I t^*} = B_P \alpha_P e^{-\alpha_P t^*} \implies L_I = B_P \alpha_P e^{(\alpha_I - \alpha_P)t^*} \tag{87}$$

Substituting (87) into (86) and simplifying gives

$$\frac{\alpha_I}{b_P} e^{\alpha_P t^*} - \alpha_P e^{\alpha_I t^*} = \alpha_I - \alpha_P, \tag{88}$$

which cannot be further simplified.

B.3 Proof of Proposition 5

Suppose

$$\exists \, \underline{t}, \bar{t} \in \underline{G}_\infty > \underline{t} : X_P(\underline{t}) > 0, X_I(\bar{t}) > 0. \tag{89}$$

If

$$(X_I(\underline{t}) + X_P(\underline{t}))^{-\eta} > e^{(r-\delta_I)(\bar{t}-\underline{t})}(X_I(\bar{t}) + X_P(\bar{t}))^{-\eta}, \tag{90}$$

then I can do better to reallocate funding from $\bar{t}$ to $\underline{t}$. Likewise, if

$$(X_I(\underline{t}) + X_P(\underline{t}))^{-\eta} < e^{(r-\delta_P)(\bar{t}-\underline{t})}(X_I(\bar{t}) + X_P(\bar{t}))^{-\eta}, \tag{91}$$

then P can do better to reallocate funding from $\underline{t}$ to $\bar{t}$. Since $\delta_I > \delta_P$, at least one of these two inequalities must obtain. Thus (89) cannot obtain in equilibrium.

It follows that there is some time t^* such that I is the sole funder for $t < t^*$ and P is the sole funder for $t > t^*$. As this suggests, t^* is a discontinuity point; let us therefore use the grid sequence G characterized by $g_n = t^* \cdot 10^{-n}$. Note that this implies that P funds the good at t^*.

As in Proposition 1, observe that each party i does best to spend such that the δ_i-discounted marginal value of spending at any time is equal to that of investing to spend at any subsequent time at which she spends. This will happen precisely when

$$X_I(t) = L_I e^{(r-\alpha_I)t}, \quad t \in [0, t^*); \tag{92}$$

$$X_P(t) = L_P e^{(r-\alpha_P)t}, \ t \in [t^*, \infty),$$

for some constants L_I, L_P.

From the budget constraints

$$\int_0^{t^*} L_I e^{-\alpha_I t} dt = B_I; \tag{93}$$

$$\int_{t^*}^{\infty} L_P e^{-\alpha_P t} dt = B_P,$$

we then have

$$L_I = B_I \alpha_I \left(1 - e^{-\alpha_I t^*}\right)^{-1}; \tag{94}$$

$$L_P = B_P \alpha_P e^{-\alpha_P t^*}.$$

Finally, observe that the spending rate must be continuous at t^*; $\lim_{t \uparrow t^*} X_I(t) = X_P(t^*)$. If the spending rate rose discontinuously at t^*, P would do better to reallocate some spending from t^* to $t^* - dt$. Likewise, if the spending rate fell discontinuously, I would do better to reallocate marginal spending from $t^* - dt$ to t^*. We

therefore have

$$L_I e^{(r-\alpha_I)t^*} = L_P e^{(r-\alpha_P)t^*}$$

$$\implies t^* = \frac{1}{\alpha_I - \alpha_P} \ln\left(\frac{L_I}{L_P}\right). \tag{95}$$

Substituting (95) into (94) and simplifying, we get

$$L_I = B_I \alpha_I + B_P \alpha_P; \tag{96}$$

$$L_P = B_P \alpha_P \left(1 + \frac{B_I \alpha_I}{B_P \alpha_P}\right)^{\frac{\alpha_P}{\alpha_I}}.$$

Finally, substituting (96) into (92) and (95), we have our final expressions for $X_I(t)$ and $X_P(t)$ and for t^* respectively.

B.4   Proof of Proposition 6

B.4.1   Preliminaries

Let $X_{P,t}(X_I)$ and $Y_{P,t}(X_I)$ denote the spending schedule and allocation, respectively, that P adopts in equilibrium given I's choice of spending schedule $X_I(t)$ (and allocation $Y_I(t)$). Let $T_P(X_I)$ denote $\{t : X_{P,t}(X_I) > 0\}$, and let $T_I(X_I)$ denote $\{t : X_I(t) > 0\}$. Let $Y(X_I) \leq B_I$ denote the budget that I allocates to spending at times $t \in T_P(X_I)$.

P will spend such that the δ_P-discounted marginal value of spending at any time $s \in T_P(X_I)$ is equal to that of investing to spend at any subsequent time $t \in T_P(X_I), t > s$. As in Proposition 1, this will happen precisely when

$$X_P(t) = L_P e^{(r-\alpha_P)t} \ \ \forall t \in T_P(X_I), \tag{97}$$

for some constant L_P. (We will ignore the irrelevant possibility of measure-zero deviations on P's part.) Furthermore, L_P will be chosen so that X_P satisfies the budget constraint

$$\int_{T_P(X_I)} e^{-rt}(X_I(t) + X_{P,t}(X_I))dt = B_P + Y(X_I). \tag{98}$$

The resulting $\{X_I(t) + X_{P,t}(X_I)\}_{t \in T_P(X_I)}$ will implement the unique δ_P-optimal allocation of resources $B_P + Y(X_I)$ across $T_P(X_I)$.

Let $\tilde{T} \subset T_P(X_I)$ denote a set of times such that

$$\int_{\tilde{T}} e^{-rt}(X_I(t) + X_{P,t}(X_I))dt = Y(X_I). \tag{99}$$

Such $\tilde{T}$ must exist, by the continuity of total resource allocation with respect to time, and that if $Y(X_I) > 0$ any such $\tilde{T}$ must have positive measure.

Suppose $Y(X_I) > 0$, and consider a spending schedule $\tilde{X}_I$ such that

$$\tilde{X}_I(t) = \begin{cases} X_I(t), & t \notin T_P(X_I); \\ X_I(t) + X_{P,t}(X_I), & t \in \tilde{T}; \\ 0, & t \in T_P(X_I)\backslash\tilde{T}. \end{cases} \tag{100}$$

P will still be able to achieve a collective spending rate of $X_I(t) + X_{P,t}(X_I)$ across $T_P(X_I)$, by spending at rate

$$X_{P,t}(\tilde{X}_I) = \begin{cases} 0, & t \notin T_P(X_I)\backslash\tilde{T}; \\ X_I(t) + X_{P,t}(X_I), & t \in T_P(X_I)\backslash\tilde{T}. \end{cases} \tag{101}$$

This collective spending rate still produces the unique δ_P-optimal allocation of resources $B_P + Y(X_I)$ across $T_P(X_I)$. Furthermore, because $\tilde{X}_I(t) = X_I(t) \ \forall t \notin T_P(X_I)$, P still prefers spending within $T_P(X_I)$ to spending outside $T_P(X_I)$. Given $\tilde{X}_I$, therefore, P does indeed best respond with the schedule $X_P(\tilde{X}_I)$ defined by (101).

$\tilde{X}_I$ thus induces the same collective spending schedule as X_I, across all t. However, $Y(\tilde{X}_I) = 0$. We have found that, for any spending schedule X_I, there is a spending schedule $\tilde{X}_I$ such that

$$Y(\tilde{X}_I) = 0 \tag{102}$$

and such that I is indifferent between X_I and $\tilde{X}_I$.

Observe by the reasoning of Proposition 1 that, given a feasible spending schedule X_I satisfying (102), I has a weakly preferred feasible spending schedule $\tilde{X}_I$, also satisfying (102), with

$$\tilde{X}_I(t) = \begin{cases} 0, & t \notin T_I(X_I); \\ \max(L_I e^{(r-\alpha_I)t}, L_P e^{(r-\alpha_P)t}), & t \in T_I(X_I) \end{cases} \tag{103}$$

for some $L_I > 0$ and for L_P as given by (98). Furthermore, if X_I is a positive-measure deviation from $\tilde{X}_I$, $\tilde{X}_I$ is strictly preferred to X_I.

That is, I always does better to shift her resource allocation from times offering her lower to times offering her higher discounted marginal utility, at least subject to the constraint that such a shift does not render her spending at a time low enough that P responds by altering his spending schedule.

Consider a feasible allocation Y_I (and corresponding spending schedule X_I) satisfying (102) and (103). We must have

$$Y_I(t) = \begin{cases} L_I e^{-\alpha_I t}, & t \in T_I(X_I) \cap [0, \bar{q}]; \\ L_P e^{-\alpha_P t}, & t \in T_I(X_I) \cap [\bar{q}, \infty), \end{cases} \tag{104}$$

for some $L_I > 0, L_P > 0$, where

$$\bar{q} \triangleq \ln\left(\frac{L_P}{L_I}\right) \Big/ (\alpha_P - \alpha_I). \tag{105}$$

Define

$$\underline{Q}(q) \triangleq \int_{T_P(X_I) \cap [0,q)} L_P e^{-\alpha_P t}, \tag{106}$$

$$\overline{Q}(q) \triangleq \int_{\tilde{T}_I(X_I) \cap [\max(\bar{q},q),\infty)} L_P e^{-\alpha_P t}.$$

Since $\underline{Q}(q)$ weakly increases in q from zero to $B_P > 0$, $\overline{Q}(q)$ weakly decreases in q from a nonnegative value to zero, and $\overline{Q}(q) - \underline{Q}(q)$ is weakly decreasing in q for all $q > 0$, there exists a (not necessarily unique) $q^* \geq 0$ such that $\underline{Q}(q^*) = \overline{Q}(q^*) \triangleq Q$. ($Q = 0$ iff I allocates nothing past $\overline{q}$.)

Fixing q^*, now consider the allocation

$$\tilde{Y}_I(t) = \begin{cases} L_P e^{-\alpha_P t}, & t \in T_P(X_I) \cap [0, q^*); \\ 0, & T_I(X_I) \cap [\max(\overline{q}, q^*), \infty); \\ Y_I(t), & \text{elsewhere.} \end{cases} \tag{107}$$

It follows from (106) that P's unique best response (up to measure-zero deviations) to a shift from Y_I to $\tilde{Y}_I$ is to shift his spending from $T_P(X_I) \cap [0, q^*)$ to $T_I(X_I) \cap [\max(\overline{q}, q^*), \infty)$, leaving his spending elsewhere unchanged; this alone maintains (97) for some L_P. It likewise follows from (106) that $\tilde{Y}_I$ is affordable for I, and that it induces the same collective allocation as Y_I. Finally, note that

$$t < \overline{q} \quad \forall t \in T_I(\tilde{X}_I),$$

$$\tilde{Y}_I(t) = L_I e^{-\alpha_I t} \quad \forall t \in T_I(\tilde{X}_I) \cap [\inf(T_P(\tilde{X}_I)), \overline{q}). \tag{108}$$

Thus, for any feasible allocation Y_I satisfying (102) and (103), there is an equally preferred feasible allocation $\tilde{Y}_I$ satisfying (102) and (108) such that I is indifferent between Y_I and $\tilde{Y}_I$.

From here we will proceed differently depending on the value of η.

B.4.2 $\eta > 1$ case

Consider a feasible allocation Y_I (and corresponding spending schedule X_I) satisfying (102) and (108). Define

$$\underline{Q}(q) \triangleq \int_{T_P(X_I) \cap [0,q)} e^{-\alpha_P t} dt, \tag{109}$$

$$\overline{Q}(q) \triangleq \int_{T_I(X_I) \cap [q,\overline{q})} e^{-\alpha_P t} dt. \tag{110}$$

Since $\underline{Q}(q)$ weakly increases in q from zero to a positive value, $\overline{Q}(q)$ weakly decreases in q from a positive value to zero, and $\overline{Q}(q) - \underline{Q}(q)$ is strictly decreasing in q for all $q > 0$, there exists a unique $q^* > 0$ such that $\underline{Q}(q^*) = \overline{Q}(q^*) \triangleq Q$. ($Q = 0$ iff I spends her entire budget before P spends any positive quantity.) Let

$$\underline{T}(X_I) \triangleq T_P(X_I) \cap [0, q^*), \tag{111}$$

$$\overline{T}(X_I) \triangleq T_I(X_I) \cap [q^*, \overline{q}).$$

(We will omit the X_I arguments to $\underline{T}$ and $\overline{T}$ when the implicit spending schedule is clear.) Define L_P as the value such that $Y_P(t) = L_P e^{-\alpha_P t} \; \forall t \in T_P(X_I)$. It follows from (102) that $Y_I(t) \geq L_P e^{-\alpha_P t} \; \forall t \in T_I(X_I)$.

If $Q > 0$, choose $\epsilon > 0$, and partition $\overline{T}$ into (not necessarily nonempty) elements

$$\overline{T}_{i,\epsilon} \triangleq \overline{T} \cap [q^* + i - \epsilon, q^* + i) \tag{112}$$

for all $i \in \epsilon \mathbb{N}$, where $\epsilon \mathbb{N}$ denotes $\{\epsilon, 2\epsilon, ...\}$. Also, define

$$\underline{t}_0 \triangleq 0,$$

$$\underline{t}_i \triangleq \min \left\{ q : \int_{\underline{T} \cap [0,q)} e^{-\alpha_P t} dt = \int_{\overline{T} \cap [q^*, q^*+i)} e^{-\alpha_P t} dt \right\}, \tag{113}$$

$$\underline{T}_{i,\epsilon} \triangleq \underline{T} \cap [\underline{t}_{i-\epsilon}, \underline{t}_i), \tag{114}$$

$$i(t, \epsilon) \triangleq i \text{ given } t \in \underline{T}_{i,\epsilon} \cup \overline{T}_{i,\epsilon}. \tag{115}$$

Let

$$S(X_I) \triangleq \left\{ s \in \overline{T}(X_I) : \lim_{\epsilon \to 0} \left(\underline{t}_{i(s,\epsilon)} - \underline{t}_{i(s,\epsilon)-\epsilon} \right) > 0 \right\}. \tag{116}$$

It follows from (113) that, for all $s \in S(X_I)$,

$$\exists \phi > 0 : \int_{\underline{T} \cap [0,\underline{t}_s)} e^{-\alpha_P t} dt = \int_{\underline{T} \cap [0,\underline{t}_s+\phi)} e^{-\alpha_P t} dt = \int_{\overline{T} \cap [q^*, s)} e^{-\alpha_P t} dt. \tag{117}$$

Given $s \in S(X_I)$, let ϕ_s denote the supremum ϕ satisfying (117). Thus, for each $s \in S(X_I)$, there is a maximal near-empty subinterval $\Phi(s) \triangleq [\underline{t}_s, \phi_s)$ of $[0, q^*)$ such that $\mu(\underline{T} \cap \Phi(s)) = 0$, where μ denotes the Lebesgue measure. Because any interval can be partitioned into at most countably many subintervals, there are at most countably many such Φ. Furthermore, for each Φ, we must have

$$\mu(\{s : \Phi(s) = \Phi\}) = 0; \tag{118}$$

otherwise the integral on the right-hand side of (117) could not be equal for all such s. Therefore $\mu(S(X_I)) = 0$.

It follows that, given any allocation Y_I satisfying (102) and (108), there is a corresponding allocation

$$\tilde{Y}_I(t) = \begin{cases} Y_I(t), & t \notin S(X_I); \\ 0, & t \in S(X_I), \end{cases} \tag{119}$$

and corresponding $\tilde{X}_I$, also satisfying (102) and (108), but for which we also have

$$S(\tilde{X}_I) = \emptyset. \tag{120}$$

Of course $X_P(\tilde{X}_I) = X_P(X_I)$ everywhere except at $S(X_I)$, and I is indifferent between Y_I and $\tilde{Y}_I$.

Retaining our formalism from (109) onward, consider an allocation Y_I (and corresponding spending schedule X_I) satisfying (102), (108), and (120) but for which $\underline{T}(X_I)) \cup \overline{T}(X_I) \neq \emptyset$. Consider the allocation

$$\tilde{Y}_I(t) = \begin{cases} Y_I(t), & t \notin \overline{T} \cup \underline{T}; \\ L_I e^{(\alpha_P - \alpha_I)(q^* + i(t,\epsilon)) - \alpha_P(\underline{t}_{i(t,\epsilon)} + \epsilon)}, & t \in \underline{T}; \\ 0, & t \in \overline{T} \end{cases} \tag{121}$$

(and corresponding spending schedule $\tilde{X}_I$). Note that

$$\overline{T}(\tilde{X}_I) = \underline{T}(\tilde{X}_I) = \emptyset. \tag{122}$$

If $\mu(\overline{T}(X_I)) = 0$ (or equivalently, $\mu(\underline{T}(X_I)) = 0$), it is clear that $\tilde{Y}_I$ is feasible and that I is indifferent between Y_I and $\tilde{Y}_I$. Let us now consider the case in which $\mu(\overline{T}(X_I)) > 0$.

To demonstrate that $\tilde{Y}_I$ is feasible, let us show that its allocation to each $\underline{T}_{i,\epsilon}$ is weakly (and indeed strictly) less than Y_I's allocation to the corresponding $\overline{T}_{i,\epsilon}$. From (113) and (114), we have

$$\int_{\underline{T}_{i,\epsilon}} e^{-\alpha_P t} dt = \int_{\overline{T}_{i,\epsilon}} e^{-\alpha_P t} dt. \tag{123}$$

Observe that $t < \underline{t}_i \ \forall t \in \underline{T}_{i,\epsilon}$ and $t \geq q^* + i - \epsilon \ \forall t \in \overline{T}_{i,\epsilon}$. Also, from (108) and (112),

$$Y_I(t) \geq L_I e^{-\alpha_I(q^* + i)} \ \ \forall t \in \overline{T}_{i,\epsilon}. \tag{124}$$

Thus (123) gives

$$\int_{\underline{T}_{i,\epsilon}} e^{-\alpha_P \underline{t}_i} dt \leq \int_{\overline{T}_{i,\epsilon}} e^{-\alpha_P(q^* + i - \epsilon)} dt \tag{125}$$

$$\implies \int_{\underline{T}_{i,\epsilon}} L_I e^{(\alpha_P - \alpha_I)(q^* + i) - \alpha_P(\underline{t}_i + \epsilon)} dt \leq \int_{\overline{T}_{i,\epsilon}} L_I e^{-\alpha_I(q^* + i)} dt$$

$$< \int_{\overline{T}_{i,\epsilon}} L_I e^{-\alpha_I t} dt. \tag{126}$$

Summing across $i \in \epsilon \mathbb{N}$, it follows that, since Y_I is feasible, $\tilde{Y}_I$ is also feasible.

Now let us show that, for sufficiently small ϵ, $\tilde{Y}_I$ is preferred to Y_I.

First, for any given ϵ, we can decompose the move from Y_I to $\tilde{Y}_I$ into a sequence of shifts from $Y_I(t)$ to $\tilde{Y}_I(t)$ for $t \in \underline{T}_{i,\epsilon} \cup \overline{T}_{i,\epsilon}$ for each i, in which she maintains the original allocation elsewhere. That is, she can shift spending back from $\overline{T}$ to $\underline{T}$ by shifting spending back from $\overline{T}_{i,\epsilon}$ to $\underline{T}_{i,\epsilon}$ for each i. Each shift will be affordable, as shown by (126). Furthermore, from (113) and (114) we see that, in equilibrium, P will (barring measure-zero deviations) respond to each shift by shifting his own allocated funds from $\underline{T}_{i,\epsilon}$ to $\overline{T}_{i,\epsilon}$, leaving his spending elsewhere unchanged; this alone will maintain condition (97) for some L_P. Finally, observe that the shift must increase flow spending and thus utility throughout $\underline{T}_{i,\epsilon}$ and decrease it throughout $\overline{T}_{i,\epsilon}$, and that discounted flow utility must monotonically decrease in t throughout both $T_P(X_I)$ and $T_I(X_I)$.

I's net utility gain from shift i is thus bounded below by

$$\frac{1}{1-\eta}\left[\int_{\underline{T}_{i,\epsilon}} e^{-\delta_I \underline{t}_i}\left(\left(L_I e^{[(\alpha_P - \alpha_I)(q^*+i) - \alpha_P(\underline{t}_i + \epsilon) + r\underline{t}_i]}\right)^{1-\eta} - \left(L_P e^{(r-\alpha_P)\underline{t}_{i-\epsilon}}\right)^{1-\eta}\right)dt \right.$$

$$\left. + \int_{\overline{T}_{i,\epsilon}} e^{-\delta_I(q^*+i-\epsilon)}\left(\left(L_P e^{(r-\alpha_P)(q^*+i)}\right)^{1-\eta} - \left(L_I e^{(r-\alpha_I)(q^*+i-\epsilon)}\right)^{1-\eta}\right)dt \right]. \tag{127}$$

From (123), the fact that $t \geq \underline{t}_{i-\epsilon} \ \forall t \in \underline{T}_{i,\epsilon}$, and the fact that $t < q^* + i \ \forall t \in \overline{T}_{i,\epsilon}$, we have

$$\int_{\overline{T}_{i,\epsilon}} e^{-\alpha_P(q^*+i)}dt \leq \int_{\underline{T}_{i,\epsilon}} e^{-\alpha_P \underline{t}_{i-\epsilon}}dt. \tag{128}$$

After rearranging (127) and making the substitution from (128), we see that I's net utility gain from shift i is further bounded below by

$$\frac{1}{\eta-1}\left(L_P e^{(r-\alpha_P)\underline{t}_{i-\epsilon}}\right)^{1-\eta} \int_{\overline{T}_{i,\epsilon}} e^{-\alpha_P(q^*+i)}dt \tag{129}$$

$$\left[\left(1 - \left(\frac{L_I}{L_P}e^{(\alpha_P - \alpha_I)(q^*+i)+(r-\alpha_P)(\underline{t}_i - \underline{t}_{i-\epsilon})-\alpha_P \epsilon}\right)^{1-\eta}\right)e^{\alpha_P \underline{t}_{i-\epsilon} - \delta_I \underline{t}_i}\right.$$

$$\left. - \left(1 - \left(\frac{L_I}{L_P}e^{(\alpha_P - \alpha_I)(q^*+i)-(r-\alpha_I)\epsilon}\right)^{1-\eta}\right)e^{(1-\eta)(r-\alpha_P)(q^*+i-\underline{t}_{i-\epsilon})-\delta_I(q^*+i-\epsilon)+\alpha_P(q^*+i)}\right].$$

By (116) and (120),

$$\lim_{\epsilon \to 0} \underline{t}_{i(t,\epsilon)} = \lim_{\epsilon \to 0} \underline{t}_{i(t,\epsilon)-\epsilon} = \underline{t}_{t-q^*} \ \ \forall t \in \overline{T}. \tag{130}$$

Also, by (112) and (115),

$$\lim_{\epsilon \to 0} i(t,\epsilon) = t - q^* \ \ \forall t \in \overline{T}. \tag{131}$$

Furthermore, $\underline{t}_{i(t,\epsilon)} - \underline{t}_{i(t,\epsilon)-\epsilon}$ is uniformly bounded by q^*, and $i(t,\epsilon) - (t - q^*)$ by $\bar{q} - q^*$, for $t \in \overline{T}$; so these two convergences are uniform throughout $\overline{T}$. It follows that

$$\lim_{\epsilon \to 0} \left[1 - \left(\frac{L_I}{L_P} e^{(\alpha_P - \alpha_I)(q^* + i(t,\epsilon)) + (r - \alpha_P)(\underline{t}_{i(t,\epsilon)} - \underline{t}_{i(t,\epsilon)-\epsilon}) - \alpha_P \epsilon} \right)^{1-\eta} \right] \tag{132}$$

$$= \lim_{\epsilon \to 0} \left[1 - \left(\frac{L_I}{L_P} e^{(\alpha_P - \alpha_I)(q^* + i(t,\epsilon)) - (r - \alpha_I)\epsilon} \right)^{1-\eta} \right] \tag{133}$$

$$= 1 - \left(\frac{L_I}{L_P} e^{(\alpha_P - \alpha_I)t} \right)^{1-\eta} \tag{134}$$

for all $t \in \overline{T}$, and that the convergence of (132) and (133) to (134) is uniform throughout $\overline{T}$.

By (102) and (108),

$$L_I e^{-\alpha_I t} > L_P e^{-\alpha_P t} \quad \forall t <\in [q^*, \bar{q}), \tag{135}$$

so $\frac{L_I}{L_P} e^{(\alpha_P - \alpha_I)t} > 1$ for all such t. By our assumption of $\eta > 1$, term (134) is positive for all $t \in \overline{T}$.

The net utility gain for I from the shift from Y_I to $\tilde{Y}_I$—the sum of (136) across $i \in \epsilon \mathbb{N}$—equals

$$\frac{1}{\eta - 1} \int_{\overline{T}} \left(L_P e^{(r - \alpha_P)\underline{t}_{i(t,\epsilon)-\epsilon}} \right)^{1-\eta} e^{-\alpha_P(q^* + i(t,\epsilon))} \tag{136}$$

$$\left[\left(1 - \left(\frac{L_I}{L_P} e^{(\alpha_P - \alpha_I)(q^* + i(t,\epsilon)) + (r - \alpha_P)(\underline{t}_{i(t,\epsilon)} - \underline{t}_{i(t,\epsilon)-\epsilon}) - \alpha_P \epsilon} \right)^{1-\eta} \right) e^{\alpha_P \underline{t}_{i(t,\epsilon)-\epsilon} - \delta_I \underline{t}_{i(t,\epsilon)}} \right.$$

$$\left. - \left(1 - \left(\frac{L_I}{L_P} e^{(\alpha_P - \alpha_I)(q^* + i(t,\epsilon)) - (r - \alpha_I)\epsilon} \right)^{1-\eta} \right) e^{(1-\eta)(r - \alpha_P)(q^* + i(t,\epsilon) - \underline{t}_{i(t,\epsilon)-\epsilon}) - \delta_I(q^* + i(t,\epsilon) - \epsilon) + \alpha_P(q^* + i(t,\epsilon))} \right] dt.$$

From (130) to (135), this converges, as $\epsilon \to 0$, to a value strictly greater than

$$\frac{L_P^{1-\eta}}{\eta - 1} \int_{\overline{T}} \left(1 - \left(\frac{L_I}{L_P} e^{(\alpha_P - \alpha_I)t} \right)^{1-\eta} \right) \left(e^{(\delta_P - \delta_I)\underline{t}_{t-q^*}} - e^{(\delta_P - \delta_I)t} \right) dt, \tag{137}$$

which is positive. Therefore the total net utility gain is positive; I strictly prefers $\tilde{Y}_I$, when constructed with a small ϵ, to Y_I.

We have shown that, if $\eta > 1$, for any allocation Y_I satisfying (102), (108), and (120), but not (122), there is a weakly preferred allocation $\tilde{Y}_I$ satisfying all four conditions, and that the preference is strict if $\mu(\overline{T}(Y_I)) > 0$.

If $\eta > 1$, given an allocation Y_I (and corresponding spending schedule X_I) satisfying (102), (108), (120), and (122), there is a time t^* (in the case of the $\tilde{X}_I$ constructed just above, equal to q^*) such that

$$X_{P,t}(X_I) = 0 \text{ for } t < t^* \text{ and } X_I(t) = 0 \text{ for } t \geq t^*. \tag{138}$$

We will now find the allocation X_I^* satisfying (138) that (uniquely, up to measure-zero deviations) maximizes I's utility in equilibrium.

Because P invests his resources until t^* and subsequently allocates them patient-optimally, we have

$$X_{P,t} = \begin{cases} 0, & t < t^*; \\ L_P e^{(r-\alpha_P)t}, & t \geq t^*, \end{cases} \tag{139}$$

where L_P satisfies

$$\int_{t^*}^{\infty} L_P e^{-\alpha_P t} dt = B_P \implies L_P = B_P \alpha_P e^{\alpha_P t^*}. \tag{140}$$

Likewise, fixing t^*, I's most preferred X_I satisfying (138) must spend I's resources impatient-optimally up to t^*. Spending impatient-optimally up to arbitrarily high t^* will not be compatible with (108), and thus not with (138), in equilibrium; P will eventually prefer spending to waiting until t^*. Nevertheless, we will find the t^* that would be optimal for I if I *could* spend impatient-optimally up to t^*, instead of eventually having to switch to a patient-optimal schedule (as in (104)). We will then see that spending impatient-optimally up to the optimal t^* is indeed compatible with (108) and thus (138) in equilibrium.

So we have

$$X_I^*(t) = \begin{cases} L_I e^{(r-\alpha_I)t}, & t < t^*; \\ 0, & t \geq t^*, \end{cases} \tag{141}$$

where L_I satisfies

$$\int_0^{t^*} L_I e^{-\alpha_I t} dt = B_I \implies L_I = \frac{B_I \alpha_I}{1 - e^{-\alpha_I t^*}}. \tag{142}$$

Given t^*, therefore, I attains utility

$$\frac{1}{1-\eta} \left[\int_0^{t^*} e^{-\delta_I t} \left(\left(B_I \alpha_I \left(1 - e^{-\alpha_I t^*} \right)^{-1} e^{(r-\alpha_I)t} \right)^{1-\eta} - 1 \right) dt \tag{143}$$

$$+ \int_{t^*}^{\infty} e^{-\delta_I t} \left(\left(B_P \alpha_P e^{\alpha_P t^*} e^{(r-\alpha_P)t} \right)^{1-\eta} - 1 \right) dt \right].$$

Simplifying and integrating this gives

$$\frac{1}{1-\eta} \left[B_I^{1-\eta} \alpha_I^{-\eta} \left(1 - e^{-\alpha_I t^*} \right)^{\eta} + \frac{(B_P \alpha_P)^{1-\eta}}{\delta_I + \alpha_P - \delta_P} e^{-\eta \alpha_I t^*} - \frac{1}{\delta_I} \right]. \tag{144}$$

From the first order condition in t^*, we find a unique maximum at

$$t^* = \ln \left(\frac{B_I \alpha_I}{B_P \alpha_P} \gamma + 1 \right) / \alpha_I. \tag{145}$$

Substituting (145) into (142) and (140), we find that the impatient-optimal allocation rate approaching t^*, and the patient-optimal allocation rate at t^*, are respectively

$$L_I e^{-\alpha_I t^*} = B_P \alpha_P / \gamma, \tag{146}$$

$$L_P e^{-\alpha_P t^*} = B_P \alpha_P.$$

Since $\gamma < 1$, spending impatient-optimally up to t^* is compatible with (108) in equilibrium, as promised. X_I^* is thus the unique optimal spending schedule among those satisfying (102), (108), (120), and (122).

The proof is completed in B.4.5.

B.4.3 $\eta = 1$ case

Follow the proof of the $\eta > 1$ case up to line (126). By the reasoning preceding (127), we can decompose the spending shift from $\overline{T}$ to $\underline{T}$, which constitutes the shift from Y_I to $\tilde{Y}_I$, into shifts from $\overline{T}_{i,\epsilon}$ to $\underline{T}_{i,\epsilon}$ for each $i \in \epsilon\mathbb{N}$; and having done so, I's net utility loss from shift i is bounded above by

$$\int_{\underline{T}_{i,\epsilon}} e^{-\delta_I \underline{t}_{i,\epsilon}} \left(\ln(L_P e^{(r-\delta_P)\underline{t}_{i-\epsilon,\epsilon}}) - \ln \left(L_I e^{[(\delta_P - \delta_I)(q^*+i) - \delta_P(\underline{t}_{i,\epsilon}+\epsilon) + r\underline{t}_{i,\epsilon}]} \right) \right) dt$$

$$+ \int_{\overline{T}_{i,\epsilon}} e^{-\delta_I(q^*+i-\epsilon)} \left(\ln \left(L_I e^{(r-\delta_I)(q^*+i-\epsilon)} \right) - \ln \left(L_P e^{(r-\delta_P)(q^*+i)} \right) \right) dt. \tag{147}$$

After rearranging, and by (128), this implies that that I's total net utility loss across i is further bounded above by

$$\int_{\overline{T}} e^{-\delta_P(q^*+i(t,\epsilon))} \left[e^{\delta_P \underline{t}_{i(t,\epsilon)-\epsilon} - \delta_I \underline{t}_{i(t,\epsilon)}} \left(\ln \left(\frac{L_P}{L_I} \right) + r(\underline{t}_{i(t,\epsilon)-\epsilon} - \underline{t}_{i(t,\epsilon)}) + (\delta_I - \delta_P)(q^* + i(t,\epsilon)) + \delta_P \epsilon \right) \right.$$

$$\left. - e^{(\delta_P - \delta_I)(q^* + i(t,\epsilon)) - \delta_I \epsilon} \left(\ln \left(\frac{L_P}{L_I} \right) + (\delta_I - \delta_P)(q^* + i(t,\epsilon)) + (r - \delta_I)\epsilon \right) \right] dt. \tag{148}$$

By the uniform convergences of (130) and (131), (148) converges to zero as $\epsilon \to 0$. Thus, for any $\ell > 0$,

$$|U_I(X_I + X_P(X_I)) - U_I(\tilde{X}_I + X_P(\tilde{X}_I))| < \ell \tag{149}$$

given any sufficiently small ϵ.

Denote the spending schedule $\tilde{X}_I$ constructed on the basis of a given $\epsilon > 0$ by $\tilde{X}_I^\epsilon$. Observe that, as $\epsilon \to 0$, $\tilde{X}_I^\epsilon(t)$ converges uniformly throughout $[0, \infty)$ to a spending schedule we might denote $\tilde{X}_I^0$, which satisfies (102), (108), (120), and (122), and such that

$$|U_I(X_I + X_P(X_I)) - U_I(\tilde{X}_I^0 + X_P(\tilde{X}_I^0))| < \ell \ \forall \ell > 0$$

$$\implies \tilde{X}_I^0 \sim_I X_I. \tag{150}$$

Let us restrict ourselves to considering spending schedules satisfying (102), (108), (120), and (122). By the reasoning of (138)–(142), the optimal spending schedule for I in this class—which we may denote X_I^*—will offer I a utility of

$$\int_0^{t^*} e^{-\delta_I t} \ln\left(\frac{B_I\delta_I}{1-e^{-\delta_I t^*}}e^{(r-\delta_I)t}\right)dt + \int_{t^*}^\infty e^{-\delta_I t}\ln\left(B_P\delta_P e^{\delta_P t^*}e^{(r-\delta_P)t}\right)dt \quad (151)$$

for some t^*. Simplifying and integrating this gives

$$\frac{e^{-\delta_I t^*}}{\delta_I}\left(\ln(B_P\delta_P) - \frac{\delta_P}{\delta_I} + \delta_I t^* + 1\right) + \frac{e^{-\delta_I t^*}-1}{\delta_I}\ln\left(\frac{1-e^{-\delta_I t^*}}{B_I\delta_I}\right) - \frac{1}{\delta_I} + \frac{r}{\delta_I^2}. \quad (152)$$

From the first order condition in t^*, we find a unique maximum at

$$t^* = \ln\left(1 + \frac{B_I\delta_I}{B_P\delta_P}e^{\frac{\delta_P}{\delta_I}-1}\right)\Big/\delta_I. \quad (153)$$

Substituting (153) into (142) and substituting both terms into (141), we obtain our expression for X_I^*.

The proof is completed in B.4.5.

B.4.4 $\eta < 1$ case

Let us pick up from what immediately precedes B.4.2.

Consider a feasible allocation Y_I (and corresponding spending schedule X_I) satisfying (102) and (108). Define

$$\underline{Q}(q) \triangleq \int_{T_I(X_I)\cap[0,q)} e^{-\alpha_P t}dt, \quad (154)$$

$$\overline{Q}(q) \triangleq \int_{T_P(X_I)\cap[q,\bar{q})} e^{-\alpha_P t}dt. \quad (155)$$

Since $\underline{Q}(q)$ weakly increases in q from zero to a positive value, $\overline{Q}(q)$ weakly decreases in q from a nonnegative value to zero, and $\overline{Q}(q) - \underline{Q}(q)$ is strictly decreasing in q for all $q > 0$, there exists a unique $q^* \geq 0$ such that $\underline{Q}(q^*) = \overline{Q}(q^*) \triangleq Q$. ($Q = 0$ iff P is the sole spender before q^*, and I is the sole spender from q^* to $\bar{q}$), for some q^*.) Let

$$\underline{T}(X_I) \triangleq T_I(X_I) \cap [0, q^*), \quad (156)$$

$$\overline{T}(X_I) \triangleq T_P(X_I) \cap [q^*, \bar{q}).$$

(We will omit the X_I arguments to $\underline{T}$ and $\overline{T}$ when the implicit spending schedule is clear.) Define L_P as the value such that $Y_P(t) = L_P e^{-\alpha_P t}$ $\forall t \in T_P(X_I)$. It follows from (102) that $Y_I(t) \geq L_P e^{-\alpha_P t}$ $\forall t \in T_I(X_I)$.

If $Q > 0$, choose $\epsilon > 0$. Partition $\overline{T}$ and define $\overline{T}_{i,\epsilon}$, $\underline{t}_i$, $\underline{T}_{i,\epsilon}$, $i(t,\epsilon)$, and $S(X_I)$ as in (112)–(116).

By the reasoning up to (120), given any allocation Y_I satisfying (102) and (108), there is a corresponding allocation $\tilde{Y}_I$ (and corresponding spending schedule $\tilde{X}_I$) also satsifying (102) and (108) but for which we also have (120), such that $\tilde{Y}_I \sim_I Y_I$.

Consider a feasible allocation Y_I (and corresponding spending schedule X_I) satisfying (102), (108), and (120) but for which $\underline{T}(X_I) \cup \overline{T}(X_I) \neq \emptyset$. Consider the allocation

$$\tilde{Y}_I(t) = \begin{cases} Y_I(t), & t \notin \overline{T} \cup \underline{T}; \\ 0, & t \in \underline{T}; \\ L_I e^{\alpha_P(\underline{t}_{i(t,\epsilon)-\epsilon}-q^*-i(t,\epsilon))-\alpha_I \underline{t}_{i(t,\epsilon)}}, & t \in \overline{T} \end{cases} \tag{157}$$

(and corresponding spending schedule $\tilde{X}_I$). Note that

$$\overline{T}(\tilde{X}_I) = \underline{T}(\tilde{X}_I) = \emptyset. \tag{158}$$

If $\mu(\overline{T}(X_I)) = 0$ (or equivalently, $\mu(\underline{T}(X_I)) = 0$), it is clear that $\tilde{Y}_I$ is feasible and that I is indifferent between Y_I and $\tilde{Y}_I$. Let us now consider the case in which $\mu(\overline{T}(X_I)) > 0$.

To demonstrate that $\tilde{Y}_I$ is feasible, let us show that its allocation to each $\overline{T}_{i,\epsilon}$ is weakly (and indeed strictly) less than Y_I's allocation to the corresponding $\underline{T}_{i,\epsilon}$. From (113) and (114), we have (123). Also, observe that $t \geq \underline{t}_{i(t,\epsilon)-\epsilon} \, \forall t \in \underline{T}_{i,\epsilon}$ and $t < q^* + i \, \forall t \in \overline{T}_{i,\epsilon}$. Also, from (108) and (112),

$$Y_I(t) \geq L_I e^{-\alpha_I \underline{t}_i} \quad \forall t \in \underline{T}_{i,\epsilon}. \tag{159}$$

Thus

$$\int_{\overline{T}_{i,\epsilon}} e^{-\alpha_P(q^*+i)} dt \leq \int_{\underline{T}_{i,\epsilon}} e^{-\alpha_P \underline{t}_i} dt \tag{160}$$

$$\implies \int_{\overline{T}_{i,\epsilon}} L_I e^{\alpha_P(\underline{t}_{i-\epsilon}-q^*-i)-\alpha_I \underline{t}_i} dt \leq \int_{\underline{T}_{i,\epsilon}} L_I e^{-\alpha_I \underline{t}_i} dt$$

$$< \int_{\underline{T}_{i,\epsilon}} L_I e^{-\alpha_I t} dt. \tag{161}$$

Summing across $i \in \epsilon \mathbb{N}$, it follows that, since Y_I is feasible, $\tilde{Y}_I$ is also feasible.

By calculations precisely analogous to those from (126) to (137)—here simply moving I's spending forward from $\underline{T}$ to $\overline{T}$, rather than the reverse—the total net utility gain for I from the shift from Y_I to $\tilde{Y}_I$ converges, as $\epsilon \to 0$, to a value strictly greater than (137) (here slightly rearranged):

$$\frac{L_P^{1-\eta}}{1-\eta} \int_{\overline{T}} \left(\left(\frac{L_I}{L_P} e^{(\alpha_P-\alpha_I)t} \right)^{1-\eta} - 1 \right) \left(e^{(\delta_P-\delta_I)\underline{t}_{t-q^*}} - e^{(\delta_P-\delta_I)t} \right) dt. \tag{162}$$

Though $\eta < 1$, this is still positive: both the coefficient outside the integral and the first product of the integral have changed sign. Therefore the total net utility gain is positive; I strictly prefers $\tilde{Y}_I$, when constructed with a small ϵ, to Y_I.

We have shown that, if $\eta < 1$, for any allocation Y_I satisfying (102), (108), and (120), but not (122), there is a weakly preferred allocation $\tilde{Y}_I$ satisfying all four conditions, and that the preference is strict if $\mu(\overline{T}(Y_I)) > 0$.

If $\eta < 1$, given an allocation Y_I (and corresponding spending schedule X_I) satisfying (102), (108), (120), and (122), there are times t_* and t^* (in the case of the $\tilde{X}_I$ constructed just above, equal to q^* and $\bar{q}$ respectively) such that

$$X_{P,t}(X_I) = 0 \text{ for } t \in [t_*, t^*) \text{ and } X_I(t) = 0 \text{ for } t \notin [t_*, t^*). \tag{163}$$

Consider a feasible allocation Y_I (and corresponding spending schedule X_I) satisfying (163) for some $t_*(X_I), t^*(X_I)$. (Where clear, we will suppress the argument). Because P, in equilibrium, allocates his resources patient-optimally across $[0, t_*) \cup [t^*, \infty)$, we have

$$X_{P,t}(X_I) = \begin{cases} 0, & t \in [t_*, t^*); \\ L_P e^{(r-\alpha_P)t}, & t \in [0, t_*) \cup [t^*, \infty), \end{cases} \tag{164}$$

where L_P satisfies

$$B_P = \int_0^{t_*} L_P e^{-\alpha_P t} dt + \int_{t^*}^{\infty} L_P e^{-\alpha_P t} dt$$

$$\implies \quad L_P = \frac{B_P \alpha_P}{1 + e^{-\alpha_P t^*} - e^{-\alpha_P t_*}}. \tag{165}$$

Likewise, I's most preferred X_I(s) satisfying (163) must spend I's resources impatient-optimally across $[t_*, t^*)$. Spending impatient-optimally for an arbitrarily long interval will not be compatible with (108), and thus not with (163), in equilibrium; P will eventually prefer spending during the interval. Nevertheless, we will find the t_*, t^* that would be optimal for I if I *could* spend impatient-optimally across an arbitrary $[t_*, t^*)$, instead of eventually having to switch to an impatient-optimal schedule (as in (104)). We will then see that the impatient-optimal schedule satisfying (163) is indeed compatible with (108) and thus (163) in equilibrium.

So we have

$$X_I(t) = \begin{cases} L_I e^{(r-\alpha_I)t}, & t \in [t_*, t^*); \\ 0, & t \in [0, t_*) \cup [t^*, \infty), \end{cases} \tag{166}$$

where L_I satisfies

$$\int_{t_*}^{t^*} L_I e^{-\alpha_I t} dt = B_I \quad \implies \quad L_I = \frac{B_I \alpha_I}{e^{-\alpha_I t_*} - e^{-\alpha_I t^*}}. \tag{167}$$

X_I therefore offers I utility

$$\left[\int_0^{t_*} e^{-\delta_I t}\left(L_P e^{(r-\alpha_P)t}\right)^{1-\eta} dt + \int_{t_*}^{t^*} e^{-\delta_I t}\left(L_I e^{(r-\alpha_I)t}\right)^{1-\eta} dt\right.$$

$$\left. + \int_{t^*}^{\infty} e^{-\delta_I t}\left(L_P e^{(r-\alpha_P)t}\right)^{1-\eta} dt - \frac{1}{\delta_I}\right] \frac{1}{1-\eta}. \tag{168}$$

Fixing t_* and L_I, it follows from (167) that

$$t^* = -\ln\left(e^{-\alpha_I t_*} - \frac{B_I \alpha_I}{L_I}\right)\Big/\alpha_I. \tag{169}$$

Substituting (169) and (165) for t^* and L_P respectively, integrating, and simplifying yields

$$\left[\left(\frac{B_P \alpha_P}{1+\left(e^{-\alpha_I t_*} - \frac{B_I \alpha_I}{L_I}\right)^{\frac{\alpha_P}{\alpha_I}} - e^{-\alpha_P t_*}}\right)^{1-\eta} \frac{1+\left(e^{-\alpha_I t_*} - \frac{B_I \alpha_I}{L_I}\right)^{\frac{\alpha_P+\delta_I-\delta_P}{\alpha_I}} - e^{-(\alpha_P+\delta_I-\delta_P)t_*}}{\alpha_P+\delta_I-\delta_P}\right.$$

$$\left. + L_I^{-\eta} B_I - \frac{1}{\delta_I}\right]\frac{1}{1-\eta}. \tag{170}$$

Differentiating with respect to t_* (and re-introducing L_P in places, for clarity) gives

$$-\frac{1}{\alpha_P+\delta_I-\delta_P}\frac{L_P^{2-\eta}}{B_P \alpha_P}\left(1+e^{-(\alpha_P+\delta_I-\delta_P)t^*} - e^{-(\alpha_P+\delta_I-\delta_P)t_*}\right) \tag{171}$$

$$+\frac{L_P^{1-\eta}}{1-\eta}\left(e^{-(\alpha_P+\delta_I-\delta_P)t_*} - \left(e^{-\alpha_I t_*} - \frac{B_I \alpha_I}{L_I}\right)^{\frac{\alpha_I-\alpha_P}{\alpha_I}(\eta-1)} e^{-\alpha_I t_*}\right).$$

The first of these two added expressions is negative. The second is also negative, as we can see from the fact that it is zero when the B_I term explicitly represented equals zero, and decreases as this term increases.

Thus, for any feasible spending schedule X_I satisfying (102), (108), (120), and (122) but not

$$t_*(X_I) = 0, \tag{172}$$

there is a strictly preferred spending schedule $\tilde{X}_I$ (with, incidentally, equal L_I) satisfying all five conditions.

I's favorite spending schedule in this class is derived—for all $\eta \neq 1$—in (139)–(146). Let us denote it by X_I^*, as done there.

B.4.5 Last steps

Letting $X_P^*(t)$ denote $X_{P,t}(X_I^*)$ and

$$Z \triangleq 1 + \frac{B_I \alpha_I}{B_P \alpha_P}\, \gamma, \tag{173}$$

it follows from (139)–(142), (145) in the $\eta \neq 1$ cases, and (153) in the $\eta = 1$ case that

$$X_I^*(t) = \begin{cases} B_I \frac{Z}{Z-1} \alpha_I e^{(r-\alpha_I)t}, & t < t^*; \\ 0, & t \geq t^* \end{cases} \tag{174}$$

and

$$X_P^*(t) = \begin{cases} 0, & t < t^*; \\ B_P Z^{\frac{\alpha_P}{\alpha_I}} \alpha_P e^{(r-\alpha_P)t}, & t \geq t^* \end{cases} \tag{175}$$

for all $\eta > 0$, where

$$t^* \triangleq \ln(Z)/\alpha_I. \tag{176}$$

X_I^* is strictly preferred to any alternative spending schedule X_I^0 that is a positive-measure deviation from X_I^* (a "PMD"). This follows straightforwardly from the constructive derivation of X_I^*, as summarized:

- For any PMD X_I^0, we can construct a PMD X_I^1 satisfying (102) such that $X_I^1 \sim_I X_I^0$.

- For any PMD X_I^1 satisfying (102):

    - If X_I^1 does not satisfy (103) almost everywhere, we can construct a spending schedule $X_I^{1'}$ satisfying (102) and (103) such that $X_I^{1'} \succ_I X_I^1$. $X_I^* \succ_I X_I^{1'}$, by the full derivation above, so $X_I^* \succ_I X_I^1 \sim_I X_I^0$.
    - If X_I^1 does satisfy (103) almost everywhere, we can construct a PMD X_I^2 satisfying (102) and (103) such that $X_I^2 \sim_I X_I^1$.

- For any PMD X_I^2 satisfying (102) and (103), we can construct a PMD X_I^3 satisfying (102) and (108) such that $X_I^3 \sim_I X_I^2$.

- For any PMD X_I^3 satisfying (102) and (108), we can construct a PMD X_I^4 satisfying (102), (108), and (120) such that $X_I^4 \sim_I X_I^3$.

- For any PMD X_I^4 satisfying (102), (108), and (120), we can construct a spending schedule X_I^5 satisfying (102), (108), (120), and (138) ($\eta \geq 1$) or (163) ($\eta < 1$) such that $X_I^5 \succsim_I X_I^4$ and $X_I^5 \neq X_I^*$.

- I strictly prefers X_I^* to all other spending schedules satisfying (102), (108), (120), and (138) ($\eta \geq 1$) or (163) ($\eta < 1$). So $X_I^* \succ_I X_I^5 \succsim_I X_I^0$.

This completes the result.

B.5 Proof of Theorem 1

Given a strategy profile σ, let $t^* \triangleq \min(\{t : B_I(\chi(\sigma)_{|t}) = 0\})$. Pick $G : t^* \in \underline{G}_\infty$. In a defection equilibrium σ^D (if one exists), $t^* < \infty$, and P spends nothing until t^*. It is then clear that, starting at t^*, P will allocate his resources patient-optimally, regardless of what I has done up to t^*. So

$$\sigma_P^D(X_{|t}) = \begin{cases} 0 & B_I(X_{|t}) > 0; \\ B_P(X_{|t})\alpha_P & B_I(X_{|t}) = 0. \end{cases} \tag{177}$$

Given this σ_P^D, a strategy σ_I is a best response iff, at every node $X_{|s}$ at which $B_I(X_{|s}) > 0$, we obtain the $t^* > s$ and $\{X_{I,t}\}_{t\in[s,t^*)}$ that maximize

$$\int_s^{t^*} e^{-\delta_I(t-s)}u(X_{I,t})dt + \int_{t^*}^\infty e^{-\delta_I(t-s)}u(X_{P,t})dt, \tag{178}$$

subject to

$$\int_s^{t^*} e^{-r(t-s)}X_{I,t}dt \leq B_I(X_{|s}), \tag{179}$$

given that

$$X_{P,t} = \begin{cases} 0 & t \in [s, t^*); \\ B_P(X_{|s})e^{r(t^*-s)}\alpha_P e^{(r-\alpha_P)(t-t^*)}, & t \geq t^*. \end{cases} \tag{180}$$

Note that, because $t^* \in \underline{G}_\infty$, we must have $X_{I,t^*} = 0$.

As in the proof of Proposition 1 (Appendix B.1), observe that I does best to spend such that the marginal value of spending at any time s is equal to that of investing to spend at any subsequent time t during which she spends, and that this will happen precisely when

$$X_{I,t} = Le^{\frac{r-\delta_I}{\eta}t}, \tag{181}$$

for some constant L, across the times $t \in [s, t^*)$ during which she will spend. Subjecting this schedule to the budget constraint, given some t^*, we have

$$\int_s^{t^*} Le^{-r(t-s)}e^{(r-\alpha_I)t}dt = B_I(X_{|s}) \tag{182}$$

$$\implies L = B_I(X_{|s})\alpha_I e^{(r-\alpha_I)s}\left(1 - e^{-\alpha_I(t^*-s)}\right)^{-1}$$

$$\implies X_{I,t} = B_I(X_{|s})\alpha_I\left(1 - e^{-\alpha_I(t^*-s)}\right)^{-1}e^{(r-\alpha_I)(t-s)}, \quad t \in [s, t^*).$$

From (178), (182), and (180), the utility I attains from spending her budget optimally by t^* is therefore

$$\left[\int_s^{t^*} e^{-\delta_I(t-s)} \left(\left(B_I(X_{|s})\alpha_I \left(1 - e^{-\alpha_I(t^*-s)} \right)^{-1} e^{(r-\alpha_I)(t-s)} \right)^{1-\eta} - 1 \right) dt \right.$$

$$\left. + \int_{t^*}^{\infty} e^{-\delta_I(t-s)} \left(\left(B_P(X_{|s})e^{r(t^*-s)}\alpha_P e^{(r-\alpha_P)(t-t^*)} \right)^{1-\eta} - 1 \right) dt \right] \cdot \frac{1}{1-\eta}, \quad \eta \neq 1; \tag{183}$$

$$\int_s^{t^*} e^{-\delta_I(t-s)} \ln \left(B_I(X_{|s})\delta_I \left(1 - e^{-\delta_I(t^*-s)} \right)^{-1} e^{(r-\delta_I)(t-s)} \right) dt$$

$$+ \int_{t^*}^{\infty} e^{-\delta_I(t-s)} \ln \left(B_P(X_{|s})e^{r(t^*-s)}\delta_P e^{(r-\delta_P)(t-t^*)} \right) dt, \quad \eta = 1. \tag{184}$$

Simplifying and integrating these terms gives

$$\frac{B_I(X_{|s})^{1-\eta}}{1-\eta} \alpha_I^{-\eta} \left(1 - e^{-\alpha_I(t^*-s)} \right)^{\eta} \tag{185}$$

$$+ \frac{B_P(X_{|s})^{1-\eta}}{1-\eta} \alpha_P^{1-\eta} \frac{1}{\alpha_P + \delta_I - \delta_P} e^{-\alpha_I \eta(t^*-s)} - \frac{1}{\delta_I(1-\eta)}, \quad \eta \neq 1;$$

$$\frac{e^{-\delta_I(t^*-s)}}{\delta_I} \left(\ln(B_P(X_{|s})\delta_P) - \frac{\delta_P}{\delta_I} + \delta_I(t^*-s) + 1 \right) \tag{186}$$

$$+ \frac{e^{-\delta_I(t^*-s)} - 1}{\delta_I} \ln \left(\frac{1 - e^{-\delta_I(t^*-s)}}{B_I(X_{|s})\delta_I} \right) - \frac{1}{\delta_I} + \frac{r}{\delta_I^2}, \quad \eta = 1.$$

By these terms' first order conditions with respect to t^*, we find a unique maximum at

$$t^* = \ln(Z(B_I(X_{|s}), B_P(X_{|s})))/\alpha_I + s, \tag{187}$$

where

$$Z(B_I(X_{|s}), B_P(X_{|s})) \triangleq 1 + \frac{B_I(X_{|s})\alpha_I}{B_P(X_{|s})\alpha_P}\gamma. \tag{188}$$

By construction, σ_I^D is I's unique best response to σ_P^D. We will now show that σ_P^D is P's unique best response to σ_I^D.

Given σ_I^D, suppose P spends $X_{P,s} \geq 0$ at some grid point $s < t^*$, and follows strategy σ_P^D elsewhere. The utility P attains across $t \geq s$ is

$$\frac{(X_{I,s} + X_{P,s})^{1-\eta} - 1}{1 - \eta} dt$$

$$+ \left[\int_{s+dt}^{t^*} e^{-\delta_P(t-s)} \left(\left((B_I(X_{|s}) - X_{I,s}dt)e^{r\,dt}\alpha_I \left(1 - e^{-\alpha_I(t^*-(s+dt))}\right)^{-1} e^{(r-\alpha_I)(t-(s+dt))} \right)^{1-\eta} - 1 \right) dt \right. \quad (189)$$

$$+ \left. \int_{t^*}^{\infty} e^{-\delta_P(t-s)} \left(\left((B_P(X_{|s}) - X_{P,s}dt)e^{r\,dt}e^{r(t^*-(s+dt))}\alpha_P e^{\frac{r-\delta_P}{\eta}(t-t^*)} \right)^{1-\eta} - 1 \right) dt \right] \cdot \frac{1}{1-\eta}, \quad \eta \neq 1;$$

$$\ln(X_{I,s} + X_{P,s})dt$$

$$+ \int_{s+dt}^{t^*} e^{-\delta_P(t-s)} \ln\left((B_I(X_{|s}) - X_{I,s}dt)e^{r\,dt}\delta_I \left(1 - e^{-\delta_I(t^*-(s+dt))}\right)^{-1} e^{(r-\delta_I)(t-(s+dt))} \right) dt \quad (190)$$

$$+ \int_{t^*}^{\infty} e^{-\delta_P(t-s)} \ln\left((B_P(X_{|s}) - X_{P,s}dt)e^{r\,dt}e^{r(t^*-(s+dt))}\delta_P e^{(r-\delta_P)(t-t^*)} \right) dt, \quad \eta = 1.$$

As our expression for $X_{I,s}$, we will use (182), with $t = s$ and

$$t^* = \ln(z)/\alpha_I + s, \quad (191)$$

$$z = Z(B_I(X_{|s}), B_P(X_{|s})). \quad (192)$$

In the integrals, we will use

$$t^* = \ln(\tilde{z})/\alpha_I + s + dt, \quad (193)$$

$$\tilde{z} = Z((B_I(X_{|s}) - X_{I,s}dt)e^{r\,dt}, (B_P(X_{|s}) - X_{P,s}dt)e^{r\,dt}). \quad (194)$$

Then we will simplify and integrate, getting

$$\frac{\left(B_I(X_{|s})\alpha_I\frac{z}{z-1} + X_{P,s} \right)^{1-\eta}}{1-\eta} dt$$

$$+ \left[\left(1 - \tilde{z}^{\frac{\delta_I-\delta_P}{\alpha_I}-1}\right)\left(B_I(X_{|s})\left(1 - \alpha_I\frac{z}{z-1}dt\right)\alpha_I\frac{\tilde{z}}{\tilde{z}-1}\right)^{1-\eta}\frac{1}{\alpha_I - \delta_I + \delta_P} \right. \quad (195)$$

$$\left. + \tilde{z}^{-\frac{\alpha_P}{\alpha_I}\eta}\left(B_P(X_{|s}) - X_{P,s}dt\right)^{1-\eta}\alpha_P^{-\eta}\right]\frac{e^{-\alpha_P\eta dt}}{1-\eta} - \frac{1}{\delta_P(1-\eta)}, \quad \eta \neq 1;$$

$$\ln\left(B_I(X_{|s})\delta_I\frac{z}{z-1} + X_{P,s} \right) dt$$

$$+ \left[\left(1 - \tilde{z}^{-\frac{\delta_P}{\delta_I}}\right)\left(\ln\left(B_I(X_{|s})\left(1 - \delta_I\frac{z}{z-1}dt\right)\delta_I\frac{\tilde{z}}{\tilde{z}-1}\right) - \frac{\delta_I}{\delta_P}\right) \right. \quad (196)$$

$$\left. + \tilde{z}^{-\frac{\delta_P}{\delta_I}}\left(\ln\left((B_P(X_{|s}) - X_{P,s}dt)\delta_P \right) + \frac{r}{\delta_I}\ln(\tilde{z}) - 1 \right) + \frac{r}{\delta_P} + r\,dt\right]\frac{e^{-\delta_P dt}}{\delta_P}, \quad \eta = 1.$$

Now we will differentiate with respect to $X_{P,s}$ (recalling that $X_{P,s}$ appears in $\tilde{z}$); divide the resulting term by dt, to rescale the payoffs from deviating from infinitesimal absolute payoffs to nonzero payoffs per unit time; and set dt to 0. We then have the rescaled payoff to an instantaneous deviation:

$$\left(\frac{B_P(X_{|s})\alpha_P}{\gamma}\right)^{-\eta}\left(\frac{\alpha_P}{\gamma(\alpha_I - \delta_I + \delta_P)} - \frac{B_I(X_{|s})}{B_P(X_{|s})(\eta - 1)}\right)z^{-\frac{\alpha_P\eta}{\alpha_I}-1} \tag{197}$$

$$+(B_P(X_{|s})\alpha_P)^{-\eta}\left(\frac{B_I(X_{|s})\alpha_P\gamma\eta}{(\eta - 1)(B_I(X_{|s})\alpha_I\gamma + B_P\alpha_P)} - 1\right)z^{-\frac{\alpha_P\eta}{\alpha_I}}$$

$$-\frac{B_P(X_{|s})^{-\eta}}{\alpha_I - \delta_I + \delta_P}\left(\frac{\alpha_P}{\gamma}\right)^{1-\eta}z^{-\eta} + \left(\alpha_I B_I(X_{|s}) + \frac{B_P(X_{|s})\alpha_P}{\gamma} + X_{P,s}\right)^{-\eta}, \; \eta \neq 1;$$

$$\gamma\frac{B_I(X_{|s})}{B_P(X_{|s})^2\delta_P}\left(2 - \frac{\delta_I}{\delta_P} - \frac{\delta_P}{\delta_I}\right)z^{-\frac{\delta_P}{\delta_I}-1} \tag{198}$$

$$-\frac{1}{B_P(X_{|s})\delta_P z} + \frac{1}{B_P(X_{|s})\delta_P z/\gamma + X_{P,s}}, \qquad\qquad \eta = 1.$$

As we can see, the total payoff impact can be split into the additional flow utility from spending at s—the last term in both expressions above—and the utility impact from adjusting I's spending schedule, and thus t^* and ultimately P's spending schedule, after s. The former is decreasing in $X_{P,s}$, and the latter is independent of $X_{P,s}$. To verify that P never wants to deviate with any positive spending rate at s, therefore, we only have to verify that the expressions above are always negative at $X_{P,s} = 0$. In the $\eta \neq 1$ case, this is equivalent to the condition that

$$\frac{1}{\gamma}\frac{\alpha_I - \alpha_P - \delta_I + \delta_P}{\alpha_I - \delta_I + \delta_P}\left(1 + \frac{B_I(X_{|s})}{B_P(X_{|s})}\frac{\alpha_I}{\alpha_P}\gamma\frac{\eta+1}{\eta-1}\frac{\alpha_I - \delta_I + \delta_P}{\alpha_I - \alpha_P - \delta_I + \delta_P}\right)z^{\eta-1-\frac{\alpha_P}{\alpha_I}\eta} \geq 1 - \frac{\alpha_P}{\alpha_I - \delta_I + \delta_P}\frac{1}{\gamma}. \tag{199}$$

In the $\eta = 1$ case, this is equivalent to the condition that

$$\frac{B_I(X_{|s})}{B_P(X_{|s})}\left(\frac{\delta_I}{\delta_P} + \frac{\delta_P}{\delta_I} - 2\right)z^{-\frac{\delta_P}{\delta_I}} \geq 1 - \frac{1}{\gamma}. \tag{200}$$

If $\eta < 1$,

$$\frac{\alpha_I - \alpha_P - \delta_I + \delta_P}{\alpha_I - \delta_I + \delta_P} > 0$$

and

$$\frac{\eta + 1}{\eta - 1}\frac{\alpha_I - \delta_I + \delta_P}{\alpha_I - \alpha_P - \delta_I + \delta_P} > 1.$$

Recall our formula for z from (192) and (188), and observe that z, and thus z to any power, is positive. It follows that the left hand side of (199) is greater than

$$\frac{1}{\gamma}\frac{\alpha_I - \alpha_P - \delta_I + \delta_P}{\alpha_I - \delta_I + \delta_P}z^{\eta-\frac{\alpha_P}{\alpha_I}\eta}. \tag{201}$$

Observe that $z > 1$ and that, when $\eta < 1$, $\eta - \frac{\alpha_P}{\alpha_I}\eta > 0$. Since $\gamma < 1$, from here it is easy to verify that condition (199) holds.

If $\eta > 1$, twice differentiate the left hand side of (199) with respect to $\frac{B_I}{B_P}$, recalling that this term appears in z but nowhere else in the expression. (I am supressing the "$X_{|s}$" argument here for clarity.) We find that the expression has a unique global minimum at

$$\frac{B_I}{B_P} = \frac{\alpha_P}{\alpha_P - \alpha_I}\frac{1}{\gamma}\Big(\frac{1}{\eta} + \frac{\eta - 1}{\eta + 1}(1 - \alpha_I + \delta_I - \delta_P)\Big). \tag{202}$$

If (202) is nonpositive, then as long as (199) holds at $\frac{B_I}{B_P} = 0$, it will hold at all $\frac{B_I}{B_P} > 0$. So we must simply evaluate (199) after substituting 0 for $\frac{B_I}{B_P}$. If (202) is positive, we must evaluate (199) after substituting (202) for $\frac{B_I}{B_P}$. In both cases, we find that the inequality holds.

If $\eta = 1$, it follows from $\delta_I > \delta_P$ that the left-hand side of (200) is positive and that the right-hand side is negative. Thus the inequality holds.

If $\eta < 1$, this game is continuous at infinity, since the range of payoffs is bounded and both parties employ positive discount rates. By the one-shot deviation principle in continuous time, therefore, σ_P^D is a best response to σ_I^D.

If $\eta \geq 1$, given a node $X_{|s}$, consider a deviation by P from σ_P^D to an alternative strategy $\tilde{\sigma}_P$—subject, as usual, to the technical restriction that

$$\tilde{\chi} \triangleq \chi(X_{|s}, (\sigma_I^D, \tilde{\sigma}_P)) \tag{203}$$

is defined, given G, for all $t \geq s$.

First, observe that σ_P^D maximizes P's forward-looking optimization problem at all nodes $X_{|t} : B_I(X_{|t}) = 0$. Deviation to $\tilde{\sigma}_P$ therefore can only offer an improvement at a node $X_{|s} : B_I(X_{|s}) > 0$.

If

$$\exists t^* : B_I(\tilde{\chi}_{|t^*}) = 0, \tag{204}$$

$\tilde{\sigma}_P$ offers P lower utility than σ_P^D. This can be seen by backward induction. First, because of the optimality of σ_P^D following t^*, a permanent deviation to $\tilde{\sigma}_P$ cannot offer P higher utility than a deviation to $\tilde{\sigma}_P$ until t^* followed by a reversion to σ_P^D. Then let

$$\bar{t} \triangleq \max_{t \in \underline{G}_\infty \cap [s,t^*)} : \chi_{P,t} > 0. \tag{205}$$

By the undesirability of one-period deviations, P prefers (a) deviating to $\tilde{\sigma}_P$ until $t - dt$ and subsequently following σ_P^D to (b) following $\tilde{\sigma}_P$ until t^*. This reasoning may be applied repeatedly until P does no spending before t^*. Deviation to a strategy $\tilde{\sigma}_P$ whose corresponding $\tilde{\chi}$ satisfies (204) is thus undesirable for P.

Consider a strategy $\tilde{\sigma}_P$ not satisfying (204). For any grid point $q > s$, by the reasoning above, P's payoff to following $\tilde{\sigma}_P$ until q, and σ_P^D subsequently, is less than $U_P(\chi(\sigma^D))$. So, denoting the continuation payoff to following strategy σ_P at grid point q (after following $\tilde{\sigma}_P$ until q) by

$$C(\sigma_P, q) \triangleq \int_q^\infty e^{-\delta_P(t-q)} u(\tilde{\chi}_{|q}, (\sigma_I^D, \sigma_P))_t) dt,$$

we have

$$C(\sigma_P^D, s) - \left(C(\tilde{\sigma}_P, s) + e^{-\delta_P q}(C(\sigma_P^D, q) - C(\tilde{\sigma}_P, q))\right) > 0 \ \ \forall q > s \qquad (206)$$

$$\implies C(\tilde{\sigma}_P, q) - C(\sigma_P^D, q) > e^{\delta_P q}(C(\tilde{\sigma}_P, s) - C(\sigma_P^D, s)) \ \ \forall q > s. \qquad (207)$$

If $\tilde{\sigma}_P$ is a profitable deviation for P at $X_{|s}$, the right-hand side is positive, so the difference in continuation payoffs as a function of q must be "fast-growing", which we will define to mean bounded below by $c_0 e^{\delta_P q}$ for some constant $c_0 > 0$. $C(\tilde{\sigma}_P, q)$ can never exced the continuation payoff for P if both parties invest all funds until q and subsequently disburse them patient-optimally, which plateaus if $\eta > 1$ and grows linearly at rate r if $\eta = 1$. (See the payoff expression from Proposition 1, substituting $B(X_{|s})e^{r(q-s)}$ for B.) For the difference in continuation payoffs to be fast-growing, therefore, $C(\sigma_P^D, q)$ must be negative and fast-growing.

The patient payoff to the defection schedule given collective budget B (see Proposition 9) can be rewritten

$$\frac{B^{1-\eta}}{1-\eta} \alpha_P^{-\eta} \frac{\alpha_I \alpha_P \eta \left(b_P + (1-b_P)\frac{\alpha_I}{\alpha_P}\gamma\right)^{1-\eta} + (\delta_I - \delta_P)^2(1-\eta)b_P^{\frac{\alpha_I+\delta_P-\delta_I}{\alpha_I}} \left(b_P + (1-b_P)\frac{\alpha_I}{\alpha_P}\gamma\right)^{-\frac{\alpha_P}{\alpha_I}\eta}}{\alpha_I \alpha_P \eta + (\delta_I - \delta_P)^2(1-\eta)} - \frac{1}{\delta_P(1-\eta)}, \quad \eta > 1; \quad (208)$$

$$\frac{1}{\delta_P}\left[\frac{(\delta_I-\delta_P)^2}{\delta_I \delta_P}\left(1 + \frac{(1-b_P)\delta_I}{b_P \delta_P}\gamma\right)^{-\frac{\delta_P}{\delta_I}} + \ln(B) + \ln\left(b_P\delta_P + (1-b_P)\delta_I\gamma\right) + \frac{\delta_I-\delta_P}{\delta_I} + \frac{r-\delta_I}{\delta_P}\right], \quad \eta = 1.$$

If $\eta > 1$, the coefficient on $B^{1-\eta}$ is negative and is bounded below across $b_P \in [0,1]$. For $C(\sigma_P^D, q)$ to be fast-growing, therefore, $B(\tilde{\chi}_{|q})^{1-\eta}$ must be fast-growing. $B(\tilde{\chi}_{|q})$ must thus be bounded above by $c_1 e^{\frac{\delta_P}{1-\eta}q}$ for some $c_1 > 0$. Because the spending rate cannot sustainably shrink more slowly than the collective budget, the discounted continuation payoff to following $\tilde{\sigma}_P$ from *any* q must likewise be bounded above, for some $c_2 > 0$, by

$$\int_q^\infty e^{-\delta_P(t-s)} \frac{(c_2 e^{\frac{\delta_P}{1-\eta}t})^{1-\eta} - 1}{1-\eta} dt = -\infty. \qquad (209)$$

If $\eta = 1$, the terms added to $\ln(B)$ are likewise bounded across b_P. For $C(\sigma_P^D, q)$ to be fast-growing, therefore, $\ln(B(\tilde{\chi}_{|q}))$ must be (again, negative and) fast-growing, so $B(\tilde{\chi}_{|q})$ must be bounded above by a function $f(q)$ that falls superexponentially to zero in q quickly enough that $\ln(f(q))$ is (negative and) fast-growing. Again, because the spending rate cannot sustainably shrink more slowly than the collective budget,

the discounted continuation payoff to following $\tilde{\sigma}_P$ from any q must be bounded above, for some $c_1, c_2 > 0$, by

$$\int_q^\infty e^{-\delta_P(t-s)} \ln\left(c_1 f(t)^{c_2}\right) dt = -\infty. \tag{210}$$

This contradicts the assumption that $\tilde{\sigma}_P$ is a profitable deviation at $X_{|s}$ from σ_P^D, whose payoff is well-defined and finite (as confirmed in Proposition 9).

We have now shown that σ^D is an equilibrium. Furthermore, the definition of defection equilibrium determines P's strategy, and we have found I's unique best response to this strategy, so σ^D must be the unique defection equilibrium.

Finally, substituting (187) into (182) at $s = 0$ gives $X_{I,t}$, and substituting (187) into (180) at $s = 0$ gives $X_{P,t}$.

B.6   Proof of Proposition 7

First, let us show that the frontier of efficient payoffs is concave.

Let $U^0 = (U_I^0, U_P^0)$ and $U^1 = (U_I^1, U_P^1)$ be two efficient payoff profiles, and let X^0 and X^1 be spending schedules which attain these payoff profiles.

The mixture spending schedule X^α, defined by $X^\alpha(t) = \alpha X^1(t) + (1 - \alpha)X^0(t)$, is feasible:

$$\int_0^\infty e^{-rt}(\alpha X^1(t) + (1 - \alpha)X^0(t))dt \tag{211}$$

$$= \alpha \int_0^\infty e^{-rt}X^1(t)dt + (1 - \alpha) \int_0^\infty e^{-rt}X^0(t)dt$$

$$= \alpha B + (1 - \alpha)B = B.$$

Furthermore, the discounted flow utility that X^α offers each player i at each time t is $e^{-\delta_i t}u(\alpha X^1(t) + (1 - \alpha)X^0(t))$. By the concavity of u, this is greater than the α-mixture of the discounted flow utilities offered by X^0 and X^1, i.e. $e^{-\delta_i t}(\alpha u(X^1(t)) + (1 - \alpha)u(X^0(t)))$.

Thus X^α offers a payoff profile that is Pareto-superior to $\alpha U^1 + (1 - \alpha)U^0$. It follows that the frontier of efficient payoffs cannot exhibit any convexities.

Because the frontier of efficient payoffs is concave, an efficient spending schedule X must maximize

$$U_a(X) \triangleq aU_I(X) + (1 - a)U_P(X) \tag{212}$$

for some weight $a \in [0, 1]$. Given efficient spending schedule X, the corresponding U_a cannot be increased by moving resources between time 0 and any other time $t \in \underline{G}_\infty$. That is,

$$U_a'(X_0) = e^{rt}U_a'(X_t) \tag{213}$$

$$\implies X_0^{-\eta} = e^{rt}(ae^{-\delta_I t} + (1 - a)e^{-\delta_P t})X_t^{-\eta}$$

for all grid points t, and thus for all t. That is, X is optimal according to time preference factor

$$\beta_a(t) = ae^{-\delta_I t} + (1 - a)e^{-\delta_P t}, \tag{214}$$

or time preference rate

$$\delta_a(t) = -\frac{\beta'(t)}{\beta(t)} = \frac{a\delta_I e^{-\delta_I t} + (1 - a)\delta_P e^{-\delta_P t}}{ae^{-\delta_I t} + (1 - a)e^{-\delta_P t}}. \tag{215}$$

As we can see, $\delta_a(0) = a\delta_I + (1 - a)\delta_P$. Therefore a is not only the weight placed on I's utility, but also the weight placed on her time preference in determining the starting time preference rate.

Let $w_a(t)$ denote the weight placed on I's time preference rate at time t, such that

$$\delta(t) = w_a(t)\delta_I + (1 - w_a(t))\delta_P. \tag{216}$$

(As we can see, $w_a(0) = a$.) Substituting (215) into (216) and rearranging, we have

$$w_a(t) = \frac{a}{a + (1-a)e^{(\delta_I - \delta_P)t}}. \tag{217}$$

Having fixed weight a to place on I's forward-looking utility, the resulting spending schedule is not time-consistent, because the resulting time preference rate is not constant. Upon reaching each grid-time $s > 0$, $aU_I(X) + (1-a)U_P(X)$ can be maximized across times $t \geq s$ by following time preference rate schedule $\delta(t - s)$ rather than $\delta(t)$ as prescribed.

However, if upon reaching s we instead place weight

$$\tilde{a} = w_a(s) \tag{218}$$

on I's forward-looking utility, the resulting time preference rate schedule is the same for $t > s$ as that prescribed at time 0 using weight a. That is,

$$\delta_{\tilde{a}}(t - s) = \delta_a(t) \quad \forall t \geq s. \tag{219}$$

We can see this by substituting (218) into (217) and the result into (214), simplifying, and differentiating:

$$\beta_{\tilde{a}}(t - s) = \frac{a \cdot e^{-\delta_I(t-s)}}{a + (1-a)e^{(\delta_I - \delta_P)s}} + \frac{(1-a)e^{(\delta_I - \delta_P)s} \cdot e^{-\delta_P(t-s)}}{a + (1-a)e^{(\delta_I - \delta_P)s}} \tag{220}$$

$$= \frac{e^{\delta_I s}}{a + (1-a)e^{(\delta_I - \delta_P)s}}(ae^{-\delta_I t} + (1-a)e^{-\delta_P t})$$

$$\implies \delta_{\tilde{a}}(t - s) = -\frac{\beta'(t-s)}{\beta(t-s)} = \frac{a\delta_I e^{-\delta_I t} + (1-a)\delta_P e^{-\delta_P t}}{ae^{-\delta_I t} + (1-a)e^{-\delta_P t}} = \delta_a(t).$$

Let $X(a)$ be the efficient spending schedule implied by weight a, and let $x(a)$ be its normalization: $x_t(a) \triangleq X_t(a)/B_t$. Given $b_I \in (0,1)$, X is a Pareto improvement to the defection schedule X^D iff $x(a)$ is a Pareto improvement to the normalized defection schedule $x^D(b_I)$:

$$U_i(X(a)) \geq U_i(X^D) \tag{221}$$

$$\iff B^{1-\eta}U_i(x(a)) + \frac{B^{1-\eta} - 1}{\delta_I(1 - \eta)} \geq B^{1-\eta}U_i(x^D) + \frac{B^{1-\eta} - 1}{\delta_I(1 - \eta)} \qquad \eta \neq 1;$$

$$U_i(x(a)) + \ln(B) \geq U_i(x^D) + \ln(B) \qquad \eta = 1,$$

$$\iff U_i(x(a)) \geq U_i(x^D),$$

with the same of course holding for strict inequalities.

Given any efficient normalized schedule $x(a)$, for some weight $a \in (0,1)$, there is some range of values $[\underline{b_I}, \bar{b_I}] \subset (0,1)$ such that $x(a)$ is a Pareto improvement on $x^D(b_I)$ iff $b_I \in [\underline{b_I}, \bar{b_I}]$. This follows directly from the inefficiency of $x^D(b_I)$ for $b_I \in (0,1)$ and the facts that

- $U_I(x^D(0)) = U_I(x(0))$,

- $U_P(x^D(0)) = U_P(x(0))$,

- $U_I(x^D(1)) = U_P(x(1))$,

- $U_P(x^D(1)) = U_P(x(1))$,

- $U_I(x^D(b_I))$ is continuous and monotonically increasing in b_I, and

- $U_P(x^D(b_I))$ is continuous and monotonically decreasing in b_I.

We can thus define

$$\underline{b_I}(a) \triangleq \arg\min_{b_I} : U_P(x(a)) \geq U_P(x^D(b_I)), \tag{222}$$

$$\overline{b_I}(a) \triangleq \arg\max_{b_I} : U_I(x(a)) \geq U_I(x^D(b_I)).$$

By construction, $[\underline{b_I}(a), \overline{b_I}(a)]$ is the range of budget proportions b_I initially belonging to I such that $x(a)$ is a Pareto improvement on x^D. As shown above, it is also the b_I-range such that $X(a)$ is a Pareto improvement on X^D. So both parties to weakly prefer cooperation to defection at $t = 0$ iff $b_I \in [\underline{b_I}(a), \overline{b_I}(a)]$.

More generally, given a strategy profile σ, both parties weakly prefer the forward-looking spending schedule $X_{[t,\infty)}(a)$ to defection at all grid points t iff

$$b_I(x_{|t}(\sigma)) \in [\underline{b_I}(w_a(t)), \overline{b_I}(w_a(t))] \ \forall t \in \underline{G}_\infty. \tag{223}$$

Given b_I, consider a strategy profile σ^* that implements a Pareto improvement $X(a)$ to $Bx^D(b_I)$, and suppose that

$$\sigma_i^*(X_{|t}) = \sigma_i^D(X_{|t}) \ \forall X_{|t} \neq X_{|t}(\sigma^*) \ \forall i. \tag{224}$$

That is, if either party defects from σ^*, they both subsequently follow the defection equilibrium. Since $b_I \in [\underline{b_I}(a), \overline{b_I}(a)]$, relation (223) holds for $t = 0$. If σ^* maintains condition (223) at all grid points t, then σ^* is an equilibrium.

Since $b_I(x_{|t}(\sigma))$, $\underline{b_I}(x_{|t}(\sigma))$, and $\overline{b_I}(x_{|t}(\sigma))$ are all continuous in t for any utility weight a and any strategy profile σ, we only need to show that σ^* can be constructed such that $b_I(x_{|t}(\sigma))$ never crosses $\underline{b_I}(x_{|t}(\sigma))$ or $\overline{b_I}(x_{|t}(\sigma))$. We will now show this, by contradiction.

Consider a time t such that $b_I(x_{|t}(\sigma^*)) \leq \overline{b_I}(x_{|t}(\sigma^*))$ but $b_I(x_{|t+dt}(\sigma^*)) > \overline{b_I}(x_{|t+dt}(\sigma^*))$. That is, suppose that, by following σ^*, a node $x_{|t}$ comes when, upon continuing to follow σ^*, I will prefer defecting to further continuing to follow σ^*. Now, specify that $\sigma_I^*(x_{|t}(\sigma^*)) = X_t(a)$. That is, define σ^* such that, at t, I contributes the entirety of the spending. Then, at $t + dt$, I strictly prefers to maintain σ^* than to defect. To see this, observe that if the forward-looking defection payoff for I at $t + dt$ were higher than the forward-looking payoff from σ^* at $t + dt$, then, at t, the payoff to spending at rate $X_t(\sigma^*)$ at t followed by defection at $t + dt$ would exceed the payoff from σ^*. But, by construction, the defection payoff for I at t is the best I can get at t given that P will not spend until $b_I = 0$, which holds in either case. It would then follow that, at t, I prefers defection to following σ^*, in contradiction to our assumption.

Likewise, consider a time t such that $b_I(x_{|t}(\sigma^*)) \geq \underline{b_I}(x_{|t}(\sigma^*))$ but $b_I(x_{|t+dt}(\sigma^*)) < \underline{b_I}(x_{|t+dt}(\sigma^*))$. That is, suppose that, by following σ^*, a node $x_{|t}$ comes when, upon continuing to follow σ^*, P will prefer defecting to further continuing to follow σ^*. Now, specify that $\sigma_P^*(x_{|t}(\sigma^*)) = X_t(a)$. That is, define σ^* such that, at t, P contributes the entirety of the spending. Then $b_I(x_{|t+dt}(\sigma^*)) > b_I(x_{|t}(\sigma^*))$. Meanwhile, because $w_a(t)$ falls with time (see (217)), $U_P(w_a(t))$ rises with time; a normalized unit of resources is allocated efficiently in a way that places ever less weight on I's forward-looking utility and ever more weight on P's. And $U_P(x^D(b_I))$ decreases in b_I (as is intuitive, and can be seen formally by differentiating the expression from Proposition 9 with respect to b_I at $B = 1$). It thus follows from the definition of $b_I(a)$ (see (222)) that $\underline{b_I}(w_a(t))$ falls over time. With $b_I(x_{|t+dt}(\sigma^*)) > b_I(x_{|t}(\sigma^*))$ and $\underline{b_I}(w_a(t + dt)) < \underline{b_I}(w_a(t))$, it follows that condition (223) is maintained at $t + dt$.

In short, if b_I gets close to the upper end of the range, σ^* can require I to contribute a larger share of flow spending, and if b_I gets close to the lower end, σ^* can require P to contribute a larger share. Having constructed σ^* such that there is no time t at which $b_I(x_{|t}(\sigma))$ crosses the necessary thresholds, (223) is always maintained, and σ^* is an equilibrium.

B.7   Proof of Proposition 8

From Proposition 4, we know that, given a warm-glow impatient funder, altruistic P is able to implement patient-optimal spending of the collective budget as long as $b_P \geq (\alpha_I - \alpha_P)/\alpha_I$. If this inequality is met, therefore, the payoff to spending patiently is simply the payoff expression from Proposition 1, with δ_P as δ.

If this inequality is not met, integrate δ_P-discounted utility given the spending rates from Proposition 4. That is, calculate

$$\int_0^{t^*} e^{-\delta_P t} u\big(B_I \alpha_I e^{(r-\alpha_I)t}\big) dt \tag{225}$$

$$+ \int_{t^*}^{\infty} e^{-\delta_P t} u\big((B_I e^{(r-\alpha_I)t^*} + B_P e^{rt^*})\alpha_P e^{(r-\alpha_P)(t-t^*)}\big) dt,$$

where $t^* = \ln\left(\frac{B_I}{B_P} \frac{\alpha_I - \alpha_P}{\alpha_P}\right)/\alpha_I > 0$.

Given a warm-glow patient funder, the discounted marginal flow utility of allocations to $t \geq 0$ for altruistic I, if only P spends at t, is

$$e^{(r-\delta_I)t} u'(X_P(t)) = (B_P \alpha_P)^{-\eta} e^{(\delta_P - \delta_I)t}) \leq (B_P \alpha_P)^{-\eta}, \tag{226}$$

where this upper bound obtains at $t = 0$. Since discounted marginal flow utility at t is decreasing in spending at t, no allocation of B_I across times can achieve I a higher payoff than her payoff from warm-glow P's patient-optimal expenditure of B_P, plus B_I times the upper bound above:

$$U_{\delta_I}(B_P, \delta_P) + B_I (B_P \alpha_P)^{-\eta}. \tag{227}$$

It follows from (13) that $t^* \to 0$ as $b_I \to 0$, i.e. as $b_P \to 1$. Furthermore, the discounted marginal flow utility for I to allocation at times $t \in [0, t^*)$ *after* allocating B_I will, by I's Euler equation, equal the marginal flow utility to allocation at $t = 0$. By (87), this will equal

$$\big(B_P \alpha_P e^{(\alpha_I - \alpha_P)t^*}\big)^{-\eta}. \tag{228}$$

The added payoff for I to I's spending B_I is thus bounded below by B_I times (228). As $t^* \to 0$, therefore, the lower bound on I's total payoff converges to the upper bound given by (227).

B.8 Proof of Theorem 2

B.8.1 P's WTP given a warm-glow impatient funder

Substitute $(1-w)B_P$ for B_P in the payoff expression from Proposition 8 where $b_P < (\alpha_I - \alpha_P)/\alpha_I$:

$$\frac{B_I^{1-\eta}}{1-\eta}\alpha_I^{1-\eta}\left[\left(\frac{1}{\delta_I-\delta_P-\alpha_I}+\frac{1}{\alpha_P}\right)\left(\frac{B_I}{(1-w)B_P}\frac{\alpha_I-\alpha_P}{\alpha_P}\right)^{\frac{\delta_I-\delta_P-\alpha_I}{\alpha_I}}-\frac{1}{\delta_I-\delta_P-\alpha_I}\right]-\frac{1}{\delta_P(1-\eta)}, \ \eta\neq 1; \quad (229)$$

$$\frac{1}{\delta_P}\left[\ln(B_I\delta_I)+\frac{r-\delta_I}{\delta_P}+\left(\frac{B_I}{(1-w)B_P}\right)^{-\frac{\delta_P}{\delta_I}}\left(\frac{\delta_I-\delta_P}{\delta_P}\right)^{\frac{\delta_I-\delta_P}{\delta_I}}\right], \qquad\qquad \eta=1.$$

This is P's payoff to spending patiently and strategically, given a warm-glow impatient funder, after paying fraction w of his budget, when b_P is small. P's payoff when the collective budget is spent according to the compromise time preference rate is given by Proposition 2, with with δ_P as δ and $b_P\delta_P+b_I\delta_I$ as $\tilde{\delta}$:

$$\frac{B^{1-\eta}}{1-\eta}\left(\frac{r\eta-r+b_P\delta_P+b_I\delta_I}{\eta}\right)^{1-\eta}\left(\left(r-b_P\delta_P-b_I\delta_I\right)\frac{\eta-1}{\eta}+\delta_P\right)^{-1}-\frac{1}{\delta_P(1-\eta)}, \ \eta\neq 1; \quad (230)$$

$$\frac{1}{\delta_P^2}\left(\delta_P\ln(B_P\delta_P+B_I\delta_I)+r-b_P\delta_P-b_I\delta_I\right), \qquad\qquad \eta=1.$$

Setting (229) equal to (230) and solving for w gives

$$1-\frac{1-b_P}{b_P}\left(\frac{\left(\frac{1}{1-b_P}\frac{\eta\alpha_I-b_P(\delta_I-\delta_P)}{\eta\alpha_I}\right)^{1-\eta}\frac{\alpha_I+\delta_P-\delta_I}{\alpha_I+\delta_P-\delta_I+b_P(\delta_I-\delta_P)\frac{\eta-1}{\eta}}-1}{1-\eta}\right)^{\frac{\alpha_I}{\alpha_I+\delta_P-\delta_I}}\left(\frac{\alpha_I-\alpha_P}{\alpha_P}\right)^{\frac{\delta_P-\delta_I}{\alpha_I+\delta_P-\delta_I}}, \ \eta\neq 1; \quad (231)$$

$$1-\frac{1-b_P}{b_P}\left(\ln\left(\frac{\delta_I-b_P(\delta_I-\delta_P)}{\delta_I-b_P\delta_I}\right)+b_P\frac{\delta_I-\delta_P}{\delta_P}\right)^{\frac{\delta_I}{\delta_P}}\left(\frac{\delta_I-\delta_P}{\delta_P}\right)^{\frac{\delta_P-\delta_I}{\delta_P}}, \qquad\qquad \eta=1.$$

This is P's willingness to pay for patient behavior given a warm-glow impatient funder when b_P is sufficiently small.

For both $\eta\neq 1$ and $\eta=1$, we can see that the WTP tends to 1 as $b_P\to 0$ by taking the limit of (231), applying L'Hôpital's Rule.

B.8.2 P's WTP given an altruistic impatient funder

Differentiating the defection payoff expression from Proposition 9 with respect to B_P, we have

$$B^{1-\eta}\frac{\alpha_I\alpha_P\eta\left(B_P+\frac{B_I\alpha_I}{\alpha_P}\gamma\right)^{-\eta}+B_P^{\frac{\delta_P-\delta_I}{\alpha_I}}(\delta_I-\delta_P)^2\left(B_P+\frac{B_I\alpha_I}{\alpha_P}\gamma\right)^{-\frac{\alpha_P}{\alpha_I}\eta-1}\left[\frac{\delta_P-\delta_I+\alpha_I}{\alpha_I}\left(B_P+\frac{B_I\alpha_I}{\alpha_P}\gamma\right)-\frac{\alpha_P}{\alpha_I}\eta B_P\right]}{\alpha_I\alpha_P\eta+(\delta_I-\delta_P)^2(1-\eta)}, \ \eta\neq 1; \quad (232)$$

$$\left(B_P\delta_P+B_I\delta_I\gamma\right)^{-1}+B_P^{\frac{\delta_P-\delta_I}{\delta_I}}(\delta_I-\delta_P)^2\left(B_P+\frac{B_I\delta_I}{\delta_P}\gamma\right)^{-\frac{\delta_P}{\delta_I}-1}\frac{B_I}{\delta_I\delta_P^2}\gamma, \ \eta=1. \quad (233)$$

Let this term, with Bb_P substituted for B_P, be denoted by $\pi(b_P)$.

Differentiating (230)—the patient payoff when the collective budget is allocated according to the compromise time preference rate—with respect to B_P, and evaluating it at $B_P = 0$, we have

$$(B_I \alpha_I)^{-\eta} \frac{\alpha_P}{\alpha_I + \delta_P - \delta_I} + \frac{1}{B_I} \frac{\alpha_I - \alpha_P}{(\alpha_I + \delta_P - \delta_I)^2} \alpha_I^{1-\eta} \quad \forall \eta > 0. \tag{234}$$

Begin from the defection equilibrium given budgets (B_I, B_P). Because P's payoff is continuous in B_P, his loss from paying fraction w of his budget converges, as $B_P \to 0$ (and thus, holding B_I fixed, as $b_P \to 0$), to

$$B_P w \pi(b_P). \tag{235}$$

His net payoff loss from joining I in allocating the collective budget according to the compromise time preference rate converges, as $B_P \to 0$ (and thus, holding B_I fixed, as $b_P \to 0$), to

$$B_P \pi(b_P) - B_P \left[(B_I \alpha_I)^{-\eta} \frac{\alpha_P}{\alpha_I + \delta_P - \delta_I} + \frac{1}{B_I} \frac{\alpha_I - \alpha_P}{(\alpha_I + \delta_P - \delta_I)^2} \alpha_I^{1-\eta} \right]. \tag{236}$$

Setting (235) equal to (236) and solving for w as a function of b_P, we have

$$\lim_{b_P \to 0} w(b_P) = \lim_{b_P \to 0} \left[1 - \frac{(B_I \alpha_I)^{-\eta} \frac{\alpha_P}{\alpha_I + \delta_P - \delta_I} + \frac{1}{B_I} \frac{\alpha_I - \alpha_P}{(\alpha_I + \delta_P - \delta_I)^2} \alpha_I^{1-\eta}}{\pi(b_P)} \right]. \tag{237}$$

If $\eta \neq 1$, observe that as $b_P \to 0$ (and thus $B_P \to 0$), the first term in the numerator of (232) approaches a constant; the factor multiplying $B_P^{\frac{\delta_P - \delta_I}{\alpha_I}}$ in the second term in the numerator approaches a constant; and the denominator and $B^{1-\eta}$ coefficient are unchanged. Because $(\delta_P - \delta_I)/\alpha_I < 0$, therefore, $|\lim_{b_P \to 0} \pi(b_P)| = \infty$ if $\eta \neq 1$.

Likewise, if $\eta = 1$, observe that as $b_P \to 0$ (and thus $B_P \to 0$), the first term of (233) approaches a constant, and the factor multiplying $B_P^{\frac{\delta_P - \delta_I}{\delta_I}}$ in the second term approaches a constant. Because $(\delta_P - \delta_I)/\delta_I < 0$, therefore, $|\lim_{b_P \to 0} \pi(b_P)| = \infty$ if $\eta = 1$.

Thus, from (237), $\lim_{b_P \to 0} w(b_P) = 1$, regardless of η.

Finally, since P's payoff to patient behavior, given budgets $(B_I, B_P(1-w))$, must be weakly higher in a Pareto-superior equilibrium than in the defection equilibrium, his willingness to pay for patient behavior must be weakly higher in a Pareto-superior equilibrium than in the defection equilibrium as well. Since the latter converges to 1 as $b_P \to 0$, the former must also.

B.9 Proof of Theorem 3

B.9.1 I's WTP given a warm-glow patient funder

One upper bound on I's WTP in this case can be found by substituting $(1-w)B_I$ for B_I in (29), setting this term equal to $U_{\delta_I}(B, \delta_P)$, and solving for w:

$$
\frac{(B_P\alpha_P)^{1-\eta}}{1-\eta}\frac{1}{\alpha_P+\delta_I-\delta_P} - \frac{1}{\delta_I(1-\eta)} + (1-w)B_I(B_P\alpha_P)^{-\eta}
$$

$$
\geq \frac{(B\alpha_P)^{1-\eta}}{1-\eta}\frac{1}{\alpha_P+\delta_I-\delta_P} - \frac{1}{\delta_I(1-\eta)}
$$

$$
\implies w \leq 1 - \frac{b_P^\eta - b_P}{1-b_P}\frac{\alpha_P}{(1-\eta)(\alpha_P+\delta_I-\delta_P)}, \qquad \eta \neq 1; \tag{238}
$$

$$
\frac{\delta_I\ln(B_P\delta_P)+r-\delta_P}{\delta_I^2} + \frac{(1-w)B_I}{B_P\delta_P}
$$

$$
\geq \frac{\delta_I\ln(B\delta_P)+r-\delta_P}{\delta_I^2}
$$

$$
\implies w \leq 1 - \frac{b_P}{1-b_P}\ln\left(\frac{1}{b_P}\right)\frac{\delta_P}{\delta_I}, \qquad \eta = 1. \tag{239}
$$

This is bounded below 1 across $b_P > 0$. (By L'Hôpital's Rule, the WTP bound approaches $\frac{\delta_I-\delta_P}{\alpha_P+\delta_I-\delta_P}$ as $b_P \to 1$, for all η. In this limit the bound is exact, by Proposition 8.)

This bound on I's WTP approaches 1 as $b_P \to 0$. A second upper bound on I's WTP, however, can be found by setting her payoff to spending the *collective* budget impatient-optimally, after paying wB_I—i.e. $U_{\delta_I}(B-wB_I, \delta_I)$—equal to $U_{\delta_I}(B, \delta_P)$ and solving for w:

$$
\frac{(B-wB_I)^{1-\eta}}{1-\eta}\alpha_I^{-\eta} - \frac{1}{\delta_I(1-\eta)} \geq \frac{(B\alpha_P)^{1-\eta}}{1-\eta}\frac{1}{\alpha_P+\delta_I-\delta_P} + \frac{1}{\delta_I(1-\eta)}
$$

$$
\implies w \leq \frac{1}{1-b_P}\left(1-\frac{\alpha_P}{\alpha_I\gamma}\right), \qquad \eta \neq 1; \tag{240}
$$

$$
\frac{\delta_I\ln((B-wB_I)\delta_I)+r-\delta_I}{\delta_I^2} \geq \frac{\delta_I\ln(B\delta_P)+r-\delta_P}{\delta_I^2}
$$

$$
\implies w \leq \frac{1}{1-b_P}\left(1-\frac{\delta_P}{\delta_I\gamma}\right), \quad \eta = 1. \tag{241}
$$

This bound approaches a value strictly below 1 as $b_P \to 0$. (The bound is also exact in this limit, by Proposition 3.)

Thus I's WTP is uniformly bounded below 1 across $b_P \in (0,1)$.

B.9.2 I's WTP given an altruistic patient funder

Set I's defection payoff from Proposition 9, with $(1-w)B_I$ substituted for B_I, equal to her payoff to collective spending under time preference rate δ_P, from Proposition 2. Solving for w yields (36).