2 Basic model

2.1 Model

Let us begin with a highly simplified model in which an agent is the sole provider of some good, and let us denote the size of the agent's budget at time $t = 0$ by B. At each moment t, we will assume that the flow utility u achieved by providing the good is an isoelastic function, parametrized by η, of the rate x at which the agent spends. That is,[1]

$$u(x(t)) = \frac{x(t)^{1-\eta}}{1-\eta}. \tag{1}$$

The agent faces a constant instantaneous real interest rate r and a constant instantaneous discount rate δ. (For a patient philanthropist, this might represent factors such as the annual risk that his funds are expropriated, that he or his fund's inheritors drift from his original values, or that the world ends; any low rate pure time preference he may have; and other trends that might roughly exponentially diminish the value of waiting to spend). We need not assume that r or δ is positive. The agent's problem is then to choose the schedule of spending rates $x(t)$ that maximizes

$$\int_0^\infty e^{-\delta t} u(x(t)) dt = \int_0^\infty e^{-\delta t} \frac{x(t)^{1-\eta}}{1-\eta} dt \tag{2}$$

subject to the constraint

$$\int_0^\infty e^{-rt} x(t) dt \leq B. \tag{3}$$

Proposition 2.1 (*The optimal individual spending schedule under constant parameters*)

Suppose an individual (philanthropic or otherwise) has isoelastic utility in spending parameterized by η, a constant discount rate δ, and a budget B, and suppose she can invest her resources at a constant interest rate r. Then the individual maximizes discounted utility by following spending schedule

$$x(t) = B\left(\frac{r\eta - r + \delta}{\eta}\right) e^{\frac{r-\delta}{\eta} t}.$$

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[1]Isoelastic utility is defined as $u(x(t)) = \ln(x)$ when $\eta = 1$, by a limit condition.

Proof: Let $y(t)$ denote the resources allocated at time 0 for investment until, followed by spending at, t. Since utility in spending is time-additive, differentiable, and concave, resource allocation y will maximize utility iff, for some constant k,

$$\frac{\partial}{\partial[y(t)]}\left[e^{-\delta t}\frac{(e^{rt}y(t))^{1-\eta}}{1-\eta}\right] = k \ \forall t, \ \eta \neq 1; \tag{4}$$

$$\frac{\partial}{\partial[y(t)]}\left[e^{-\delta t}\ln(e^{rt}y(t))\right] = k \ \forall t, \ \eta = 1.$$

Taking the derivative and rearranging, we have

$$y(t) = k^{\frac{-1}{\eta}}e^{\frac{r-r\eta-\delta}{\eta}t}. \tag{5}$$

Subjecting this resource allocation to the budget constraint, we have

$$\int_0^\infty k^{\frac{-1}{\eta}}e^{\frac{r-r\eta-\delta}{\eta}t}dt = B; \tag{6}$$

$$k = \left(B\left(\frac{r\eta - r + \delta}{\eta}\right)\right)^{-\eta}. \tag{7}$$

Substituting (4) into (2), and observing that $x(t) = e^{rt}y(t)$, we have

$$x(t) = B\left(\frac{r\eta - r + \delta}{\eta}\right)e^{\frac{r-\delta}{\eta}t} \tag{8}$$

as desired. ∎

Corollary 2.2 (*The payoff under constant parameters*)

Following the optimal spending schedule $x(t)$, as given in Proposition 2.1, produces a payoff of

$$\frac{\eta\left(\frac{B}{\eta}\right)^{1-\eta}}{(1-\eta)(r\eta - r + \delta)^\eta}, \ \eta \neq 1;$$

$$\frac{\delta\ln(B\delta) + r - \delta}{\delta^2}, \ \eta = 1.$$

This can be calculated straightforwardly from the integral $\int_0^\infty e^{-\delta t}u(x(t))dt$. ∎

2.2 Discussion

The above model is motivated in this paper by the scenario in which a philanthropist is the sole provider of some public good. So far, it is equivalent to an infinite-horizon consumption-smoothing model under certainty, assuming either (a) no future outside income or (b) complete capital markets. (Note that the assumption of complete capital markets renders this problem the same as the problem one faces with no outside income. Given certainty and complete markets, someone with future income can borrow against her entire income stream, and B can represent current assets plus the present value of future income.) Nevertheless, given the centrality of the underlying relationship described

above to the analysis of patient philanthropy below, let us now take a moment to note three of its relevant features.

First, and most importantly: In the context of a simple consumption-smoothing model, the optimal spending rate is highly sensitive to the discount rate. The patient, that is, should spend slowly. As we can see from (8) at $t = 0$, it is always optimal to spend at proportional rate $\frac{r\eta - r + \delta}{\eta}$. In particular, if $\eta = 1$, the spending rate should equal δ. For instance, philanthropists who are funding idiosyncratic projects with no other present or future funders, who discount future impacts at 0.1% per year, and who are confident that the world (or their philanthropic projects) will not soon be brought to an end, should spend only 0.1% of their budgets per year.

Second: As one might expect, whether outflows are increasing, constant, or decreasing in time depends on whether $r - \delta$ is greater than, equal to, or less than zero. Furthermore, observe the exponent on

$$x'(t) = B\Big(\frac{r\eta - r + \delta}{\eta}\Big)\Big(\frac{r - \delta}{\eta}\Big)e^{\frac{r-\delta}{\eta}t}. \tag{9}$$

If $r - \delta > 0$, the *rate of increase* in spending is also increasing with time, and if $r - \delta < 0$, $\lim_{x\to\infty} x(t) = 0$. It follows that we have no edge cases in which one's assets should be expected to grow or shrink asymptotically to a positive size. The $r = \delta$ steady state is unstable. If a fund should grow, it should not stop growing, even once it has grown very large.

Third: If $r\eta - r + \delta \leq 0$, we appear to be led to the conclusion that it is always preferable to invest than to spend. This is, in other words, the circumstance which gives rise to Koopmans' (1965, 1967) "paradox of the indefinitely postponed splurge". For the purposes of this paper, rather than broach the subject of infinite ethics, we will assume that this circumstance does not hold. Note that it does not whenever $\eta > 1$, $r > 0$, and $\delta \geq 0$. That is, under the reasonable assumption that $\eta > 1$, agents can be arbitrarily patient without approaching paradoxical territory.

3 Model with varying parameters

3.1 General Markov-process model

The basic consumption-smoothing model introduced above, according to which patient philanthropists should spend slowly, assumes that the interest rate, discount rate, and flow utility achieved by a given expenditure rate are constant over time. As noted in the introduction, however, some patient philanthropists advocate for a higher spending rate on the grounds that they expect these parameters to vary. We will account for these complications as generally as possible by introducing a finite set S of states s, each of which comes with a state-specific interest rate $r(s)$, discount rate $\delta(s)$, and scale parameter $h(s)$, such that the flow utility from spending at rate x in state s is given by

$$u_s(x) = h(s)\frac{x^{1-\eta}}{1 - \eta}. \tag{10}$$

Instantaneous transition probabilities among the states will be given by a transition matrix T.

Note that this is equivalent to a model in which the *price* of the good being provided is a state-dependent value $p(s)$, with $h(s) \triangleq p(s)^{\eta-1}$.

Relatedly, note that as long as h is constant, the optimal spending schedule does not depend on h. More generally, because h is by construction multiplicatively separable from the rest of our formula for $u(x)$, rescaling $h(s)$ for all s amounts to a change of units, and does not affect the relative value of spending across periods.

Finally, note that if h is a martingale, the optimal spending schedule is the same as if h is constant. In this case, whatever value h may take at some time t, its expected value at all subsequent periods is equal to its observed value at t. It follows that the expected value of marginal spending at any given time is unchanged relative to the expected value of marginal spending at any other time.

Resources will be divided optimally between spending and investing, in a given state, if the marginal value of spending equals the expected marginal value of investing.[2] The expected marginal value of investing from a given state, in turn, is equal to the expected marginal value of money across subsequent states. And because resources in a given state will be split between spending and investing, each of which will have equal marginal value, the expected value of money in each state is equal to the marginal value of additional spending in that state. We can therefore say that the agent spends optimally in each state if the agent's intertemporal Euler equation is satisfied—that is, if the marginal value of spending equals the expected marginal value of investing to spend in the next period.

The isoelasticity assumption conveniently guarantees that the optimal proportion of funds to spend in state s (or, in continuous time, the optimal rate at which to spend while in state s)—which we will denote $x(s)$—depends only on η and on the features of s. In particular, it does not depend on the absolute size of the fund. To see this, observe that if the fund increases by some proportion m, and the state-contingent spending policy stays the same, the marginal value of spending in each state falls by proportion ηm. The marginal value of spending in each state will thus still equal the expected marginal value of spending in the next period; both quantities will fall by the same proportion.

More formally:

Proposition 3.1 (*The optimal spending policy given varying parameters in discrete time*)

Suppose an individual has isoelastic utility in spending parameterized by η and a state-dependent discount rate $\delta(s)$, and suppose she can invest her resources at a state-dependent interest rate $r(s)$. Suppose also that movement in the state space S is given by transition matrix T. Then the individual

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[2]The isoelastic functional form assumes infinite marginal utility when the spending rate is zero, so the agent will never quite find himself in a corner solution of spending nothing or spending his entire budget.

maximizes discounted utility by following spending policy $x(s)$ such that, for all $i \in \{1, ..., |S|\}$,

$$h(s_i)(x(s_i))^{-\eta} = e^{r(s_i)-\delta(s_i)} \sum_{j=1}^{|S|}[T_{i,j}h(s_j)(e^{r(s_i)}(1-x(s_i))x(s_j))^{-\eta}].$$

The optimal spending policy is now given by $|S|$ equations with $|S|$ unknown variables (the $x(s_i)$), and it will generally be well-defined.

For ease of exposition, the above model is set in discrete time. To find the optimal policy in continuous time, observe that the agent spends at an optimal *rate* in each state if *the marginal value of spending in that state equals the expected marginal value of investing resources for the subsequent state*, when it arrives. We can find the latter by defining the transition matrix T such that $T_{i,j}$ is the instantaneous probability of a transition to j from i, with $T_{i,i} = 0 \; \forall i$ and $\Sigma_j T_{i,j}$, not necessarily summing to 1 for any i. Now, at any time in state i, the instantaneous probability of a transition to any other state is $\sigma(s_i) \triangleq \Sigma_j T_{i,j}$, and the probability density that the subsequent transition takes place t units of time into the future is given by $\sigma(s_i)e^{-\sigma(s_i)t}$.

To find the expected value of investing marginal resources for the next state, whenever it arrives, we can therefore integrate over the possible transition times. Setting this expected value equal to the value of increasing the spending rate while in i, we have

$$h(s_i)(x(s_i))^{-\eta} = \int_0^\infty \sigma(s_i)e^{-\sigma(s_i)t}\left(e^{(r(s_i)-\delta(s_i))t} \sum_j \left[\frac{T_{i,j}}{\sigma(s_i)}h(s_j)\left(e^{(r(s_i)-x(s_i))t}x(s_j)\right)^{-\eta}\right]\right)dt. \quad (11)$$

Simplifying this expression, we have the following:

Proposition 3.2 (*The optimal spending policy given varying parameters in continuous time*)

Suppose an individual has isoelastic utility in spending parameterized by η and a state-dependent discount rate $\delta(s)$, and suppose she can invest her resources at a state-dependent interest rate $r(s)$. Suppose also that movement in the state space S is given by Poisson processes with transition rates from s_i to s_j given by matrix T. Then the individual maximizes discounted utility by following spending policy $x(s)$ such that, for all $i \in \{1, ..., |S|\}$,

$$h(s_i)(x(s_i))^{-\eta} = \frac{\sum_j \left[T_{i,j}h(s_j)(x(s_j))^{-\eta}\right]}{\sigma(s_i) - r(s_i) + \delta(s_i) + r(s_i)\eta - x(s_i)\eta}.$$

3.2   Simplification with $\eta = 1$

The above setup lets us find the optimal spending policy numerically, and perhaps sheds some light on the shape of the problem. If $\eta = 1$, however—that is, if impact is logarithmic in spending—there is an analytic solution, offering perhaps a simpler and more transparent look into how the optimal spending schedule depends on the variables involved.

Proposition 3.3 (*The optimal spending policy given varying parameters in continuous time, where* $\eta = 1$)

Suppose an individual has logarithmic utility in spending and a state-dependent discount rate $\delta(s)$, and suppose she can invest her resources at a state-dependent interest rate $r(s)$. Suppose also that movement in the state space S is given by Poisson processes with transition rates from s_i to s_j given by matrix T. Then the individual maximizes discounted utility by following the spending policy given by matrix equation

$$\vec{x} = \left(I_{|S|} - H \circ \mathcal{T} \right)^{-1} \vec{\sigma},$$

where $\vec{x}$ is the $|S|$-vector of inverse spending proportions with $x_i = \frac{1}{x(s_i)}$, $\vec{\sigma}$ is the $|S|$-vector of inverse instantaneous transition probabilities with $\sigma_i = \frac{1}{\sigma(s_i) + \delta(s_i)}$, H is the $|S| \times |S|$ "scaling matrix" with $H_{i,j} = \frac{h(s_j)}{h(s_i)}$, and $\mathcal{T}$ is the $|S| \times |S|$ "relative transition probability matrix" with $\mathcal{T}_{i,j} = \frac{T_{i,j}}{\sigma(s_i) + \delta(s_i)}$.

The result follows from substituting $\eta = 1$ into the expression in Proposition 3.2, so that we have $|S|$ linear equations of the form

$$\frac{1}{x(s_i)} = \frac{1}{\sigma(s_i) + \delta(s_i)} \left(\sum_j \left[T_{i,j} \frac{h(s_j)}{h(s_i)} \frac{1}{x(s_j)} \right] + 1 \right), \tag{12}$$

and rearranging. ∎

3.3 Expected payoffs

Proposition 3.4 (*The expected payoff given varying parameters in continuous time*)

Let

$$a_i \triangleq 1/(\sigma(s_i) + \delta(s_i) + (x(s_i) - r(s_i))(1 - \eta)).$$

If $\eta \neq 1$, he expected payoff v_i by state s_i from following a spending policy $x(s)$ is given by

$$\vec{v} = B^{1-\eta}(I_{|S|} - A \circ T)^{-1}\vec{z},$$

where A is an $|S| \times |S|$ matrix with $A_{i,j} \triangleq a_i$ and $\vec{z}$ is an $|S|$-vector with

$$z_i \triangleq h(s_i) \frac{x(s_i)^{1-\eta}}{(1-\eta)} a_i.$$

If $\eta = 1$, the expected payoff v_i by state s_i from following a spending policy $x(s)$ is given by

$$\vec{v} = (I_{|S|} - A \circ T)^{-1}(\vec{z} + \vec{w_\delta} \ln B),$$

where a_i and A are as defined above, $\vec{z}$ is an $|S|$-vector with

$$z_i \triangleq \left(\frac{r(s_i) - x(s_i) + \delta(s_i) \ln x(s_i)}{\delta(s_i)} + \frac{\sigma(s_i)h(s_i)(x(s_i) - r(s_i))}{\delta(s_i)(\sigma(s_i) + \delta(s_i))} \right) a_i + \frac{\sigma(s_i)}{\sigma(s_i) - r(s_i) + x(s_i) + \delta(s_i)},$$

and $\vec{w_\delta}$ is an $|S|$-vector with $w_{\delta,i} \triangleq \sigma(s_i)a_i/\delta(s_i)$.

The result follows straightforwardly from two observations. First, observe that, in each state s_i, the expected payoff must equal the utility that will accrue from spending at rate $x(s_i)$ until the state transition, plus the expected payoff following the transition. Second, observe that multiplying one's budget by some constant m multiplies spending at all times by m.

If $\eta \neq 1$, multiplying one's budget by m thereby multiplies flow utility at all times, and thus the payoff, by $m^{1-\eta}$. Given a starting budget of 1 and a transition time of t, for example, the continuation payoff at t will equal $e^{(r(s_i)-x(s_i))(1-\eta)t}$ multiplied by the continuation payoff in the new state given a budget at t of 1. Our payoffs must therefore satisfy the $|S|$ equations given by

$$v(s_i) = B^{1-\eta} \int_0^\infty \sigma(s_i) e^{-\sigma(s_i)t} \Big[\int_0^t e^{-\delta(s_i)q} h(s_i) \frac{(x(s_i)e^{(r(s_i)-x(s_i))q})^{1-\eta}}{1-\eta} dq \tag{13}$$

$$+ e^{-\delta(s_i)t} \sum_{j=1}^{|S|} \frac{T_{i,j}}{\sigma(s_i)} e^{(r(s_i)-x(s_i))(1-\eta)t} v(s_j) \Big] dt.$$

If $\eta = 1$, multiplying one's budget by m thereby adds $\ln m$ to flow utility at all times. We can therefore find the payoff given budget B by finding the payoff v_1 given budget 1 and adding it to the expected discounted value v_B of a stream of payoffs of size $\ln B$. These two terms will respectively satisfy

$$v_1(s_i) = \int_0^\infty \sigma(s_i) e^{-\sigma(s_i)t} \Big[\int_0^t e^{-\delta(s_i)q} h(s_i) \ln(x(s_i)e^{(r(s_i)-x(s_i))q}) dq \tag{14}$$

$$+ e^{-\delta(s_i)t} \sum_{j=1}^{|S|} \frac{T_{i,j}}{\sigma(s_i)} (e^{(r(s_i)-x(s_i))t} + v(s_j)) \Big] dt$$

and

$$v_B(s_i) = \int_0^\infty \sigma(s_i) e^{-\sigma(s_i)t} \Big[\int_0^t e^{-\delta(s_i)q} \ln B \, dq + e^{-\delta(s_i)t} \sum_{j=1}^{|S|} \frac{T_{i,j}}{\sigma(s_i)} v_B(s_j) \Big] dt \tag{15}$$

Simplifying and rearranging these expressions gives the results. ∎