Supplement: full proof of Proposition 2

Let $y_w^n \equiv A_w^n L_w^n$, so that $Y_w = \sum_n (y_w^n)^{1+\alpha}$, and let $y_w \equiv \sum_n y_w^n$ and $y \equiv \inf_w y_w$. Let $\mathbf{L}_w^n$ denote $\int_0^w L_v^n dv$, and observe that it is continuous. Let $\mathbf{L} \equiv \sum_n \mathbf{L}^n$. When subscripted with a set P rather than a number, $\mathbf{L}_P^{(n)} \equiv \int_P L_w^{(n)} dw$. Likewise, $\mathbf{y}_P^n \equiv \int_P A_w^n L_w^n dw$ and $\mathbf{y}_P \equiv \sum_n y_P^n$.

Lemma 2: Discrete approximation of effective labor. As in Lemma 1, assume that there is only one worker allocating her labor across an interval open below and closed above, and for simplicity let the interval be $(0, 1]$. For each $m \in \mathbb{N}$, let $\{I_{j,m}\}_{j=1}^{2^m}$ be the partition $I_{j,m} = ((j-1)2^{-m},\ j2^{-m}]$ and define the step approximation

$$
\begin{aligned}
L_w^{(m)} &\equiv \frac{1}{|I_{j,m}|} \int_{I_{j,m}} L_v \, dv \quad \text{for } w \in I_{j,m}, \\
H_w^{(m)} &\equiv \int_0^w (w-v)^{-\beta} L_v^{(m)} \, dv,
\end{aligned}
$$

and $A_w^{(m)}$, $y_w^{(m)}$ likewise. Then:

1. $L_{(\cdot)}^{(m)} \to L_{(\cdot)}$ in $\mathrm{L}^2(0,1]$ and $L_w^{(m)} \to L_w$ a.e.
2. For all w, $H_w^{(m)} \to H_w$ and so $A_w^{(m)} \to A_w$.
3. Consequently, $y_w^{(m)} \to y_w$ a.e.

Moreover, for all m and all $w \in (0, 1]$,

$$
H_w^{(m)} \leq (1-2\beta)^{-\frac{1}{2}} \left\| L_{(\cdot)} \right\|_{\mathrm{L}^2} w^{\frac{1}{2}-\beta}.
$$

Proof. The functions $L_{(\cdot)}^{(m)}$ are conditional expectations of $L_{(\cdot)}$. Since $L_{(\cdot)} \in \mathrm{L}^2(0,1]$, the martingale convergence theorem implies $L_{(\cdot)}^{(m)} \to L_{(\cdot)}$ in L^2. The Lebesgue differentiation theorem implies $L_{(\cdot)}^{(m)} \to L_{(\cdot)}$ pointwise a.e. This proves (1).

By the Cauchy–Schwarz inequality,

$$
\left| H_w^{(m)} - H_w \right| \leq \left(\int_0^w (w-v)^{-2\beta} dv \right)^{1/2} \left(\int_0^w \left| L_v^{(m)} - L_v \right|^2 dv \right)^{1/2}.
$$

Because $\beta < 1/2$, the first integral is finite; and since $L_{(\cdot)}^{(m)} \to L_{(\cdot)}$ in L^2, the second term tends to zero, so $H_w^{(m)} \to H_w$. Continuity of $(\cdot)^\gamma$ yields (2).

Combining (1) and (2) gives (3). The final bound follows again from Cauchy–Schwarz—

$$H_w^{(m)} \leq \left(\int_0^w (w-v)^{-2\beta} \, dv \right)^{1/2} \left\| L_{(\cdot)}^{(m)} \right\|_{\mathrm{L}^2} = (1-2\beta)^{-\frac{1}{2}} \left\| L_{(\cdot)}^{(m)} \right\|_{\mathrm{L}^2} w^{\frac{1}{2}-\beta}$$

—and $\left\| L_{(\cdot)}^{(m)} \right\|_{\mathrm{L}^2} \leq \| L_{(\cdot)} \|_{\mathrm{L}^2}$. □

Lemma 3: Binwise discrete approximation of effective labor. Maintain the setup and notation of Lemma 2. For each m and each $w \in (0,1]$, define the discrete right-endpoint map

$$\bar{w}_m(w) \;\equiv\; 2^{-m} \lceil 2^m w \rceil \in \left\{ 2^{-m}, 2 \cdot 2^{-m}, \ldots, 1 \right\},$$

so that $w \in I_{j,m}$ implies $\bar{w}_m(w) = j2^{-m}$. Define the *binwise-discrete* effective labor and learning stocks:

$$\tilde{y}_w^{(m)} \;\equiv\; y_{\bar{w}_m(w)}^{(m)} \qquad \text{and} \qquad \tilde{H}_w^{(m)} \;\equiv\; H_{\bar{w}_m(w)}^{(m)}.$$

Then $\tilde{y}_w^{(m)} \to y_w$ a.e.

More precisely, there exists a finite constant C_β (depending only on β) such that for all m and all $0 < w' \leq w \leq 1$,

$$\left| H_w^{(m)} - H_{w'}^{(m)} \right| \;\leq\; C_\beta \, \|L\|_{\mathrm{L}^2(0,1]} \, |w - w'|^{\frac{1}{2}-\beta}, \tag{18}$$

and so for almost every w,

$$\left| \tilde{y}_w^{(m)} - y_w^{(m)} \right| \;\longrightarrow\; 0, \qquad \text{so that} \qquad \tilde{y}_w^{(m)} \to y_w \text{ a.e.} \tag{19}$$

Proof. Step 1 (uniform Hölder modulus in w for $H_{(\cdot)}^{(m)}$). Fix $0 < w' \leq w \leq 1$ and write $\Delta \equiv w - w' > 0$. Split

$$H_w^{(m)} - H_{w'}^{(m)} = \int_0^{w'} \left[(w-v)^{-\beta} - (w'-v)^{-\beta} \right] L_v^{(m)} \, dv \;+\; \int_{w'}^{w} (w-v)^{-\beta} L_v^{(m)} \, dv \equiv T_1 + T_2.$$

For T_2, Cauchy–Schwarz gives

$$|T_2| \le \Big(\int_{w'}^{w} (w-v)^{-2\beta}\,dv \Big)^{1/2} \|L^{(m)}_{(\cdot)}\|_{\mathrm{L}^2} = \Big(\frac{\Delta^{1-2\beta}}{1-2\beta} \Big)^{1/2} \|L^{(m)}_{(\cdot)}\|_{\mathrm{L}^2}.$$

Using $\|L^{(m)}_{(\cdot)}\|_{\mathrm{L}^2} \le \|L_{(\cdot)}\|_{\mathrm{L}^2}$ yields $|T_2| \le (1-2\beta)^{-1/2}\|L_{(\cdot)}\|_{\mathrm{L}^2}\Delta^{\frac{1}{2}-\beta}$.

For T_1, use the pointwise bound

$$\big|(w-v)^{-\beta} - (w'-v)^{-\beta}\big| \le (w'-v)^{-\beta}\,\mathbf{1}_{\{w'-v\le\Delta\}} \;+\; \beta\,\Delta\,(w'-v)^{-\beta-1}\,\mathbf{1}_{\{w'-v>\Delta\}},$$

which follows from monotonicity of $t \mapsto t^{-\beta}$ and the mean value theorem on (Δ,∞). Applying Cauchy–Schwarz to each region and computing the resulting kernel integrals gives $|T_1| \le C'_\beta\|L^{(m)}_{(\cdot)}\|_{\mathrm{L}^2}\Delta^{\frac{1}{2}-\beta}$ for some finite C'_β depending only on β. Again using $\|L^{(m)}_{(\cdot)}\|_{\mathrm{L}^2} \le \|L_{(\cdot)}\|_{\mathrm{L}^2}$ proves (18) with $C_\beta \equiv C'_\beta + (1-2\beta)^{-1/2}$.

Step 2 (within-bin replacement $w \mapsto \bar{w}_m(w)$). Fix $w \in (0,1]$ and set $\bar{w} \equiv \bar{w}_m(w)$. Then $0 \le \bar{w} - w \le 2^{-m}$ and, by construction, $L^{(m)}_{\bar{w}} = L^{(m)}_w$ because $L^{(m)}_{(\cdot)}$ is constant on each bin. Thus

$$\tilde{y}^{(m)}_w - y^{(m)}_w = L^{(m)}_w\Big(\big(H^{(m)}_{\bar{w}}\big)^\gamma - \big(H^{(m)}_w\big)^\gamma\Big).$$

Since $\gamma \in (0,1)$ and $a, b \ge 0$ implies $|a^\gamma - b^\gamma| \le |a-b|^\gamma$, we obtain

$$\big|\tilde{y}^{(m)}_w - y^{(m)}_w\big| \le L^{(m)}_w\,\big|H^{(m)}_{\bar{w}} - H^{(m)}_w\big|^\gamma.$$

Using (18) and $\bar{w} - w \le 2^{-m}$ gives

$$\big|\tilde{y}^{(m)}_w - y^{(m)}_w\big| \le L^{(m)}_w\,\Big(C_\beta\|L_{(\cdot)}\|_{\mathrm{L}^2}\,2^{-m(\frac{1}{2}-\beta)}\Big)^\gamma.$$

By Lemma 2(1), $L^{(m)}_w \to L_w$ a.e., hence for such w the sequence $\{L^{(m)}_w\}^{(m)}$ is eventually bounded. Because $\gamma(\frac{1}{2}-\beta) > 0$, the right-hand side tends to 0, proving (19) a.e.

Step 3 (convergence to the continuous y_w). Lemma 2(3) gives $y^{(m)}_w \to y_w$ for almost all w. Combining with (19) yields $\tilde{y}^{(m)}_w \to y_w$ a.e. □

Lemma 4: Existence and characterization of the optimal allocation of a worker

over an interval. Let

$$\mathcal{A} \equiv \left\{ L_{(\cdot)} \in \mathrm{L}^2(0,1] \;:\; L_w \geq 0,\; \int_0^1 L_w \, dw = 1 \right\}$$

denote the set of admissible labor allocations after dropping the left-continuity requirement. Given $L_{(\cdot)} \in \mathcal{A}$, let $\Phi \equiv \operatorname{ess\,inf}_{w \in (0,1]} \; y_w$. Then:

1. There exists $L^*_{(\cdot)} \in \mathcal{A}$ with corresponding $\Phi^* = \sup_{L_{(\cdot)} \in \mathcal{A}} \Phi$.
2. Any maximizer $L^*_{(\cdot)}$ satisfies $y^*_w = \Phi^*$ a.e.
3. The maximizer is unique up to null sets and is given by

$$L^*_w = (1-\beta)\, w^{-\beta}, \qquad w \in (0,1].$$

Proof. Step 1 (discrete maximization). For each $m \in \mathbb{N}$, partition $(0,1]$ into intervals $I_j^{(m)} = ((j-1)2^{-m}, j2^{-m}]$, and define the simplex

$$\Delta^{(m)} \equiv \left\{ x \in \mathbb{R}_+^{2^m} \;:\; \sum_{j=1}^{2^m} x_j = 1 \right\}.$$

For $x \in \Delta^{(m)}$, define the discrete learning stock and effective labor

$$h_k^{*(m)}(x) \equiv \sum_{j=1}^{k} \left(\frac{k-j+1}{2^m} \right)^{-\beta} x_j, \qquad y_k^{(m)}(x) \equiv \left(h_k^{*(m)}(x) \right)^{\gamma} 2^m x_k,$$

and the discrete objective

$$y^{(m)}(x) \equiv \min_{1 \leq k \leq 2^m} y_k^{(m)}(x).$$

Since $y^{(m)}(\cdot)$ is continuous on the compact $\Delta^{(m)}$, there is a maximizer $x^{*(m)} \in \Delta^{(m)}$.

Step 2 (equalization at the discrete optimum). Given an allocation x, suppose

$$\exists k \;:\; y_k^{(m)}(x) > y^{(m)}(x).$$

Because $y_k^{(m)}(x)$ increases continuously in x_j for $j \leq k$, there is an $\varepsilon > 0$ such that

$y_k^{(m)}(\tilde{x}) > y^{(m)}(x)$ for any $\tilde{x}$ with (i) $\tilde{x}_k > x_k - \varepsilon$ and (ii) $\tilde{x}_j \geq x_j \; \forall j < k$.

Also, if $k \neq 2^m$, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $H_j^{(m)}(\tilde{x}) > H_j^{(m)}(x) \; \forall j > k$, and thus $y_j^{(m)}(\tilde{x}) > y_j^{(m)}(x) \; \forall j > k$, for all $\tilde{x}$ with (i) $\tilde{x}_{j'} \geq x_{j'} \; \forall j' < k$, (ii) $\tilde{x}_k = x_k(1-\varepsilon)$, (iii) $\tilde{x}_{k+1} = x_{k+1} + (1-\delta)\varepsilon x_k$, and (iv) $\tilde{x}_j \geq x_j \; \forall j > k+1$.

Choose such a ε, δ and let

$$\tilde{x}_j = \begin{cases} x_j + \delta \varepsilon x_k/(k-1), & j < k; \\ (1-\varepsilon)x_j, & j = k; \\ x_j + (1-\delta)\varepsilon x_k, & j = k+1; \\ x_j, & j > k+1 \end{cases}$$

(omitting the last two cases if $k = 2^m$). Since $y_j^{(m)}(\tilde{x}) > y_j^{(m)}(x)$ for all $j \neq k$ and $y_k^{(m)}(\tilde{x}_k) > y^{(m)}(x)$, we have $y^{(m)}(\tilde{x}) > y^{(m)}(x)$.

Step 3 (Uniform L^2 bound for the discrete equal-effective-labor solutions). Let $L_{(\cdot)}^{*(m)}$ be the step-function density associated to $x^{*(m)}$: that is, $L_w^{*(m)} \equiv 2^m x_k^{*(m)}$ for $w \in ((k-1)2^{-m}, k2^{-m}]$. Then $\{\|L_{(\cdot)}^{*(m)}\|_{L^2(0,1]}\}_{m\geq 1}$ is uniformly bounded; equivalently,

$$\sup_{m\geq 1} \; 2^m \sum_{k=1}^{2^m} \left(x_k^{*(m)}\right)^2 \; < \; \infty.$$

Proof. Define

$$M_k^{(m)} \equiv 2^m x_k^{*(m)} \left(\frac{k}{2^m}\right)^{\beta}, \qquad k = 1, \dots, 2^m,$$

so $2^m x_k^{*(m)} = M_k^{(m)}(k/2^m)^{-\beta}$. Also define

$$\overline{M}^{(m)} \equiv \max_{1\leq k\leq 2^m} M_k^{(m)}, \qquad \underline{M}^{(m)} \equiv \min_{1\leq k\leq 2^m} M_k^{(m)}.$$

Step 3.1 (Factorization of $h_k^{(m)}$ for $k \geq 2$). Fix $k \geq 2$. Using $x_j^{*(m)} = 2^{-m} M_j^{(m)} (j/2^m)^{-\beta}$,

$$\begin{aligned} h_k^{(m)}(x^{*(m)}) &= \sum_{j=1}^{k} \left(\frac{k-j+1}{2^m} \right)^{-\beta} x_j^{*(m)} \\ &= 2^{-m} \sum_{j=1}^{k} \left(\frac{k-j+1}{2^m} \right)^{-\beta} \left(\frac{j}{2^m} \right)^{-\beta} M_j^{(m)} \\ &= \left(\frac{k}{2^m} \right)^{1-2\beta} \cdot J_k^{(m)}, \end{aligned}$$

where

$$J_k^{(m)} \equiv \frac{1}{k} \sum_{j=1}^{k} \left(1 - \frac{j-1}{k} \right)^{-\beta} \left(\frac{j}{k} \right)^{-\beta} M_j^{(m)}.$$

Step 3.2 (Uniform bounds on $J_k^{(m)}$ in terms of $\underline{M}^{(m)}$ and $\overline{M}^{(m)}$). Because $\beta \in (0,1)$, the weight function $w(s) \equiv ((1-s)s)^{-\beta}$ is integrable on $(0,1)$. Moreover, the discrete weights

$$\omega_{j,k} \equiv \left(1 - \frac{j-1}{k} \right)^{-\beta} \left(\frac{j}{k} \right)^{-\beta}$$

satisfy $\omega_{j,k} \geq 1$ for all $1 \leq j \leq k$ (since each factor is ≥ 1). Therefore, for every m and $k \geq 2$,

$$J_k^{(m)} = \frac{1}{k} \sum_{j=1}^{k} \omega_{j,k} M_j^{(m)} \geq \frac{1}{k} \sum_{j=1}^{k} 1 \cdot \underline{M}^{(m)} = \underline{M}^{(m)}. \tag{20}$$

On the other hand, since $\omega \in \mathrm{L}^1(0,1]$, the Riemann sums $\frac{1}{k} \sum_{j=1}^{k} \omega_{j,k}$ are bounded above uniformly in $k \geq 2$; thus there exists $C_\beta < \infty$ such that

$$\frac{1}{k} \sum_{j=1}^{k} \omega_{j,k} \leq C_\beta \qquad \text{for all } k \geq 2.$$

Hence, for every m and $k \geq 2$,

$$J_k^{(m)} = \frac{1}{k} \sum_{j=1}^{k} \omega_{j,k} M_j^{(m)} \leq \overline{M}^{(m)} \cdot \frac{1}{k} \sum_{j=1}^{k} \omega_{j,k} \leq C_\beta \, \overline{M}^{(m)}. \tag{21}$$

Step 3.3 (Use equalization, including $k = 1$, to bound $\overline{M}^{(m)}$ and $\underline{M}^{(m)}$). By definition of $x^{*(m)}$, the quantities

$$y_k^{(m)}(x^{*(m)}) \equiv \left(h_k^{(m)}(x^{*(m)})\right)^{\gamma} 2^m x_k^{*(m)}$$

are equal across $k = 1, \ldots, 2^m$; denote their common value by $y^{*(m)}$.

First consider $k = 1$. Then $h_1^{(m)}(x^{*(m)}) = ((1)/2^m)^{-\beta} x_1^{*(m)}$ and $2^m x_1^{*(m)} = M_1^{(m)}((1)/2^m)^{-\beta}$, so

$$y^{*(m)} = y_1^{*(m)}\left(x^{*(m)}\right) = \left(M_1^{(m)}\right)^{1+\gamma}. \tag{22}$$

Now fix any $k \geq 2$. Using Step 1 and $2^m x_k^{*(m)} = M_k^{(m)}(k/2^m)^{-\beta}$,

$$y^{*(m)} = \left(\left(\frac{k}{2^m}\right)^{1-2\beta} J_k^{(m)}\right)^{\gamma} M_k^{(m)}\left(\frac{k}{2^m}\right)^{-\beta} = M_k^{(m)}\left(J_k^{(m)}\right)^{\gamma}\left(\frac{k}{2^m}\right)^{\gamma(1-2\beta)-\beta}.$$

Since $\gamma(1 - 2\beta) - \beta = 0$,

$$y^{*(m)} = M_k^{(m)}\left(J_k^{(m)}\right)^{\gamma} \qquad \text{for all } k \geq 2. \tag{23}$$

Choose $k_{\max} \in \{1, \ldots, 2^m\}$ attaining $\overline{M}^{(m)}$. If $k_{\max} = 1$, then (22) gives $\overline{M}^{(m)} = M_1^{(m)} = (y^{*(m)})^{\frac{1}{1+\gamma}}$. If $k_{\max} \geq 2$, then combining (23) with (20) yields

$$y^{*(m)} = \overline{M}^{(m)}(J_{k_{\max}}^{(m)})^{\gamma} \geq \overline{M}^{(m)}(\underline{M}^{(m)})^{\gamma}.$$

In either case we have the uniform inequality

$$y^{*(m)} \geq \overline{M}^{(m)}(\underline{M}^{(m)})^{\gamma}. \tag{24}$$

Similarly, choose $k_{\min} \in \{1, \ldots, 2^m\}$ attaining $\underline{M}^{(m)}$. If $k_{\min} = 1$, then (22) gives $\underline{M}^{(m)} = M_1^{(m)} = (y^{*(m)})^{\frac{1}{1+\gamma}}$. If $k_{\min} \geq 2$, then combining (23) with (21) yields

$$y^{*(m)} = \underline{M}^{(m)}(J_{k_{\min}}^{(m)})^{\gamma} \leq \underline{M}^{(m)}(C_\beta \overline{M}^{(m)})^{\gamma}.$$

In either case we have

$$y^{*(m)} \leq \underline{M}^{(m)}(C_\beta \overline{M}^{(m)})^{\gamma}. \tag{25}$$

Combining (24) and (25) gives

$$\overline{M}^{(m)}(\underline{M}^{(m)})^{\gamma} \leq y^{*(m)} \leq \underline{M}^{(m)}(C_{\beta}\overline{M}^{(m)})^{\gamma},$$

hence

$$\left(\frac{\overline{M}^{(m)}}{\underline{M}^{(m)}}\right)^{1-\gamma} \leq C_{\beta}^{\gamma}.$$

Therefore $\overline{M}^{(m)}/\underline{M}^{(m)}$ is bounded uniformly in m.

Finally, use the unit-mass constraint:

$$1 = \sum_{k=1}^{2^m} x_k^{\star(m)} = 2^{-m} \sum_{k=1}^{2^m} M_k^{(m)} \left(\frac{k}{2^m}\right)^{-\beta} \geq 2^{-m}\, \underline{M}^{(m)} \sum_{k=1}^{2^m} \left(\frac{k}{2^m}\right)^{-\beta}.$$

We now bound the last sum from below. Since $x \mapsto x^{-\beta}$ is decreasing on $(0, 1]$, for each $k \geq 2$ we have

$$x^{-\beta} \leq \left(\frac{k-1}{2^m}\right)^{-\beta} \quad \text{for all } x \in \left(\frac{k-1}{2^m}, \frac{k}{2^m}\right].$$

Hence

$$\int_{(k-1)/2^m}^{k/2^m} x^{-\beta}\, dx \leq 2^{-m}\left(\frac{k-1}{2^m}\right)^{-\beta}.$$

Summing over $k = 2, \dots, 2^m$ gives

$$\int_{1/2^m}^{1} x^{-\beta}\, dx \leq 2^{-m} \sum_{r=1}^{2^m-1} \left(\frac{r}{2^m}\right)^{-\beta}.$$

Since $\beta < 1$,

$$\int_{1/2^m}^{1} x^{-\beta}\, dx = \frac{1 - 2^{-m(1-\beta)}}{1-\beta}.$$

Therefore,

$$\sum_{r=1}^{2^m-1} \left(\frac{r}{2^m}\right)^{-\beta} \geq \frac{2^m\left(1 - 2^{-m(1-\beta)}\right)}{1-\beta}.$$

Because $1 - 2^{-m(1-\beta)} \geq 1 - 2^{-(1-\beta)} > 0$ for all $m \geq 1$, we obtain

$$\sum_{k=1}^{2^m} \left(\frac{k}{2^m}\right)^{-\beta} \geq \frac{c_\beta}{1-\beta}\, 2^m, \qquad c_\beta \equiv 1 - 2^{-(1-\beta)}.$$

Substituting this bound back into the unit-mass constraint yields

$$1 \geq 2^{-m}\, \underline{M}^{(m)} \cdot \frac{c_\beta}{1-\beta}\, 2^m = \frac{c_\beta}{1-\beta}\, \underline{M}^{(m)},$$

so

$$\underline{M}^{(m)} \leq \frac{1-\beta}{c_\beta}.$$

This provides a uniform upper bound on $\underline{M}^{(m)}$ independent of m.

The uniform ratio bound then implies $\overline{M}^{(m)}$ is bounded above uniformly in m as well.

Step 3.4 (Conclude the uniform L^2 bound). For every m and k,

$$2^m x_k^{*(m)} = M_k^{(m)} \Big(\frac{k}{2^m}\Big)^{-\beta} \leq \overline{M}^{(m)} \Big(\frac{k}{2^m}\Big)^{-\beta} \leq C\Big(\frac{k}{2^m}\Big)^{-\beta},$$

with C independent of m. Squaring and summing,

$$\|L_{(\cdot)}^{*(m)}\|_{\mathrm{L}^2(0,1]}^2 = \sum_{k=1}^{2^m} (2^m x_k^{*(m)})^2\, 2^{-m} \leq C^2 \sum_{k=1}^{2^m} \left(\frac{k}{2^m}\right)^{-2\beta} 2^{-m},$$

and the right-hand side is a Riemann sum for $\int_0^1 w^{-2\beta}\, dw < \infty$ since $\beta < \frac{1}{2}$. This proves $\sup_m \|L_{(\cdot)}^{*(m)}\|_{\mathrm{L}^2(0,1]} < \infty$, i.e. $\sup_m 2^m \sum_k (x_k^{*(m)})^2 < \infty$. □

Step 4 (compactness and extraction of a weak limit). Fix $m \in \mathbb{N}$ and let $w_k \equiv k2^{-m}$ and $I_{k,m} \equiv (w_{k-1}, w_k]$ for $k = 1, \dots, 2^m$. Let $x^{*(m)} = (x_1^{*(m)}, \dots, x_{2^m}^{*(m)})$ be an optimizer of the m-bucket problem, and define the associated staircase density

$$L_w^{*(m)} \equiv 2^m x_k^{*(m)} \quad \text{for } w \in I_{k,m}.$$

Assume (as established in Step 3) that the sequence $\{L^{*(m)}\}_{m\geq 1}$ is bounded in $\mathrm{L}^2(0,1]$, i.e. there exists $B < \infty$ such that $\|L^{*(m)}\|_{\mathrm{L}^2} \leq B$ for all m.

Since $\mathrm{L}^2(0,1]$ is reflexive, there exists a subsequence (not relabeled) and a function $L^*_{(\cdot)} \in \mathrm{L}^2(0,1]$ such that $L^{*(m)}_{(\cdot)} \rightharpoonup L^*_{(\cdot)}$ weakly in $\mathrm{L}^2(0,1]$. Moreover, $L^*_{(\cdot)}$ is admissible: because $1 \in \mathrm{L}^2(0,1]$, weak convergence implies $\int_0^1 L^{*(m)}_w \, dw \to \int_0^1 L^*_w \, dw$, and since each $\int_0^1 L^{*(m)}_w = 1$, we have $\int_0^1 L^*_w dw = 1$. Finally, $L^*_w \geq 0$ a.e. because the cone $\{f \in \mathrm{L}^2 : f \geq 0 \text{ a.e.}\}$ is convex and weakly closed. After extracting a further subsequence if needed, we may also assume that the optimal discrete value $y^{*(m)}$ converges: $y^{*(m)} \to y^* \in [0,\infty)$.

Step 5 (the weak limit equalizes continuous effective labor a.e.). Let $y^*_{(\cdot)}$ denote the workflow effective labor schedule induced by $L^*_{(\cdot)}$ (etc). We will now prove that, for some constant y,

$$y^*_w = y \quad \text{a.e.}$$

Step 5.1 (Uniform approximation of stocks at gridpoints). For each m and each $k \leq 2^m$, define the piecewise-constant kernel

$$\tilde{K}_{k,m,v} \equiv \begin{cases} (w_k - w_{j-1})^{-\beta}, & v \in I_{j,m} \text{ with } j \leq k, \\ 0, & v > w_k. \end{cases}$$

Then, by construction of $L^{*(m)}_{(\cdot)}$,

$$\int_0^1 \tilde{K}_{k,m,v} \, L^{*(m)}_v \, dv = \sum_{j=1}^k (w_k - w_{j-1})^{-\beta} \int_{I_{j,m}} L^{*(m)}_v \, dv = \sum_{j=1}^k (w_k - w_{j-1})^{-\beta} x^{*(m)}_j.$$

Denote the continuous learning stock induced by $L^{*(m)}_{(\cdot)}$ by $H^{*(m)}_{(\cdot)}$, and the discrete learning stock induced by $x^{*(m)}$ (or equivalently $L^{*(m)}_{(\cdot)}$) by $h^{*(m)}_{(\cdot)}$. Observe that $h^{*(m)}_k = \int_0^1 \tilde{K}_{k,m,v} L^{*(m)}_v \, dv$.

Also define the true kernel $K_{k,v} \equiv (w_k - v)^{-\beta} \mathbf{1}_{\{v \leq w_k\}}$, so that $H^{*(m)}_{w_k} = \int_0^1 K_{k,v} L^{*(m)}_v \, dv$. Hence

$$H^{*(m)}_{w_k} - h^{*(m)}_k = \int_0^1 \left(K_{k,v} - \tilde{K}_{k,m,v}\right) L^{*(m)}_v \, dv.$$

By Cauchy–Schwarz,

$$\left|H^{*(m)}_{w_k} - h^{*(m)}_k\right| \le \|K_k - \tilde{K}_{k,m}\|_{\mathrm{L}^2(0,1]}\,\|L^{*(m)}_{(\cdot)}\|_{\mathrm{L}^2(0,1]}.$$

A direct estimate of $\|K_k - \tilde{K}_{k,m}\|_{\mathrm{L}^2}$ yields the uniform bound

$$\sup_{m\ge 1}\ \sup_{1\le k\le 2^m} \frac{\|K_k - \tilde{K}_{k,m}\|_{\mathrm{L}^2(0,1]}}{2^{-m(\frac{1}{2}-\beta)}} \le C_\beta < \infty,$$

for a constant C_β depending only on β (and finite because $\beta < 1/2$). Therefore, since $\|L^{*(m)}_{(\cdot)}\|_2 \le B$ for some finite B by Step 3,

$$\max_{1\le k\le 2^m} \left|H^{*(m)}_{w_k} - h^{*(m)}_k\right| \le C_\beta B\, 2^{-m(\frac{1}{2}-\beta)} \equiv \varepsilon_m \quad \text{with } \varepsilon_m \to 0.$$

Step 5.2 (Discrete equalization and an L^1-almost-constant proxy). Recall that by Step 2,

$$y^{*(m)}_k \equiv y^{*(m)} \quad \forall k.$$

Now define the piecewise-constant proxy

$$g^{(m)}_w \equiv \left(H^{*(m)}_{w_k}\right)^\gamma \cdot 2^m x^{*(m)}_k \quad \text{for } w \in I_{k,m}.$$

For $w \in I_{k,m}$ we have

$$|g^{(m)}_w - y^{*(m)}| = 2^m x^{*(m)}_k \left|\left(H^{*(m)}_{w_k}\right)^\gamma - (h^{*(m)}_k)^\gamma\right|.$$

Since $\gamma \in (0,1)$, $x \mapsto x^\gamma$ is γ-Hölder on $\mathbb{R}_+$ and

$$|a^\gamma - b^\gamma| \le |a-b|^\gamma \quad \forall a, b \ge 0.$$

Thus, using the stock bound from Step 5.1,

$$|g^{(m)}_w - y^{*(m)}| \le 2^m x^{*(m)}_k \left|H^{*(m)}_{w_k} - h^{*(m)}_k\right|^\gamma \le 2^m x^{*(m)}_k\, \varepsilon_m^\gamma.$$

Integrating over $(0,1]$ and using $|I_{k,m}| = 2^{-m}$ and $\sum_k x_k^{*(m)} = 1$ yields

$$\|g^{(m)}_{(\cdot)} - y^{*(m)}\|_{\mathrm{L}^1(0,1]} = \sum_{k=1}^{2^m} \int_{I_{k,m}} |g^{(m)}_w - y^{*(m)}|\, dw \le \sum_{k=1}^{2^m} (2^{-m})(2^m x_k^{*(m)})\varepsilon_m^\gamma = \varepsilon_m^\gamma \to 0.$$

Hence $g^{(m)} \to y^{*(m)}$ in L^1, and along our subsequence $y^{*(m)} \to y^*$ (from Step 4) we have

$$g^{(m)}_{(\cdot)} \to y^* \quad \text{in } \mathrm{L}^1(0,1].$$

In particular, for every bounded measurable test function φ,

$$\int_0^1 \varphi_w g^{(m)}_w \, dw \to y^* \int_0^1 \varphi_w \, dw.$$

Step 5.3 (Replace $g^{(m)}_{(\cdot)}$ by the corresponding continuous expression $G^{(m)}_{(\cdot)}$ under testing). Define

$$G^{(m)}_w = \big(H^{*(m)}_w\big)^\gamma L^{*(m)}_w.$$

Fix $\varphi \in C([0,1])$. We will show that

$$\int_0^1 \varphi_w g^{(m)}_w \, dw - \int_0^1 \varphi_w G^{(m)}_w \, dw \to 0.$$

Writing the left integral cellwise and using $2^m x_k^{*(m)} = L^{*(m)}_w$ for $w \in I_{k,m}$,

$$\int_0^1 \varphi_w \, g^{(m)}_w dw = \sum_{k=1}^{2^m} \int_{I_{k,m}} \varphi_w \big(H^{*(m)}_{w_k}\big)^\gamma L^{*(m)}_w \, dw,$$

so the difference equals

$$\sum_{k=1}^{2^m} \int_{I_{k,m}} \varphi_w \Big(\big(H^{*(m)}_{w_k}\big)^\gamma - \big(H^{*(m)}_w\big)^\gamma \Big) L^{*(m)}_w \, dw.$$

By Cauchy–Schwarz, for any $L_{(\cdot)} \in \mathrm{L}^2$ and any $0 < w' < w \le 1$,

$$|H_w - H_{w'}| = \Big| \int_0^1 \Big((w-v)^{-\beta} \mathbf{1}_{v \le w} - (w'-v)^{-\beta} \mathbf{1}_{v \le w'} \Big) L_v \, dv \Big| \le C_\beta \|L_{(\cdot)}\|_2 \, |w - w'|^{\frac{1}{2}-\beta},$$

for a constant C_β depending only on β; in particular, $H^{*(m)}_{(\cdot)}$ is uniformly Hölder on $[0,1]$ with exponent $\frac{1}{2}-\beta$ and constant $C_\beta B$. Since $\gamma \in (0,1)$ and $|a^\gamma - b^\gamma| \le |a-b|^\gamma$, for $w \in I_{k,m}$ we have

$$\left|\left(H^{*(m)}_{w_k}\right)^\gamma - \left(H^{*(m)}_{w}\right)^\gamma\right| \le \left|H^{*(m)}_{w_k} - H^{*(m)}_{w}\right|^\gamma \le (C_\beta B)^\gamma\, 2^{-m\gamma(\frac{1}{2}-\beta)}.$$

So

$$\begin{aligned}\left|\int_0^1 \varphi_w\, g^{(m)}_w dw - \int_0^1 \varphi_w\, G^{(m)}_w dw\right| &\le \|\varphi\|_\infty (C_\beta B)^\gamma\, 2^{-m\gamma(\frac{1}{2}-\beta)} \int_0^1 L^{*(m)}_w\, dw \\ &= \|\varphi\|_\infty (C_\beta B)^\gamma\, 2^{-m\gamma(\frac{1}{2}-\beta)} \to 0.\end{aligned}$$

Step 5.4 (Pass to the weak limit: $\int\ \varphi\, G^{(m)} \to \int\ \varphi\, y^*$). Fix $\varphi \in C([0,1])$. We will show

$$\int_0^1 \varphi_w \big(H^{*(m)}_w\big)^\gamma L^{*(m)}_w\, dw \to \int_0^1 \varphi_w \big(H^*_w\big)^\gamma L^*_w\, dw.$$

Write the difference as

$$\int_0^1 \varphi_w\Big(\big(H^{*(m)}_w\big)^\gamma - \big(H^*_w\big)^\gamma\Big) L^{*(m)} dw \;+\; \int_0^1 \varphi_w \big(H^*_w\big)^\gamma \big(L^{*(m)}_w - L^*_w\big)\, dw \equiv \mathrm{A}^{(m)} + \mathrm{B}^{(m)}.$$

Term $\mathrm{B}^{(m)}$. Since $\beta < 1/2$ and $L^*_{(\cdot)} \in \mathrm{L}^2$, we have $H^*_{(\cdot)} \in C((0,1])$ and $H^*_w \lesssim \|L^*_{(\cdot)}\|_2\, w^{\frac{1}{2}-\beta}$ as $w \downarrow 0$, hence $\varphi_{(\cdot)}\big(H^*_{(\cdot)}\big)^\gamma \in \mathrm{L}^2(0,1]$. Because $L^{*(m)}_{(\cdot)} \rightharpoonup L^*_{(\cdot)}$ weakly in L^2, it follows that $\mathrm{B}^{(m)} \to 0$.

Term $\mathrm{A}^{(m)}$. Fix $\delta \in (0,1)$ and split

$$\mathrm{A}^{(m)} = \int_0^\delta (\cdots)\, dw + \int_\delta^1 (\cdots)\, dw \equiv \mathrm{A}^{(m)0}_\delta + \mathrm{A}^{(m)1}_\delta.$$

On $[\delta,1]$, the family $\left\{H^{*(m)}_{(\cdot)}\right\}$ is equicontinuous and uniformly bounded: for $w \in [\delta,1]$,

$$|H^{*(m)}_w| \le \|L^{*(m)}_{(\cdot)}\|_2 \|(w-(\cdot))^{-\beta}\mathbf{1}_{\{(\cdot)<w\}}\|_2 \le B\, C_{\beta,\delta},$$

and similarly for increments. Moreover, for each fixed $w > 0$, the kernel $(w - (\cdot))^{-\beta}\mathbf{1}_{\{(\cdot)<w\}} \in \mathrm{L}^2(0,1]$, hence weak convergence implies $H^{*(m)}_w \to H^*_w$ pointwise on

$(0,1]$. By Arzelà–Ascoli on $[\delta,1]$, this implies uniform convergence:

$$\sup_{w\in[\delta,1]} |H_w^{*(m)} - H_w^*| \to 0, \qquad \sup_{w\in[\delta,1]} \Big|\big(H_w^{*(m)}\big)^\gamma - \big(H_w^*\big)^\gamma\Big| \to 0.$$

Therefore,

$$|\mathrm{A}_\delta^{(m)1}| \le \|\varphi\|_\infty \sup_{w\in[\delta,1]} \Big|\big(H_w^{*(m)}\big)^\gamma - \big(H_w^*\big)^\gamma\Big| \int_\delta^1 L_w^{*(m)}\, dw \to 0 \quad \text{as } m\to\infty.$$

For the small-w part, use the bound (valid for any $L_{(\cdot)} \in \mathrm{L}^2$) $H_w \le C_\beta \|L_{(\cdot)}\|_2 w^{\frac{1}{2}-\beta}$, so for $w \in (0,\delta)$,

$$\big(H_w^{*(m)}\big)^\gamma \le (C_\beta B)^\gamma\, w^{\gamma(\frac{1}{2}-\beta)}, \qquad \big(H_w^*\big)^\gamma \le (C_\beta \|L_{(\cdot)}^*\|_2)^\gamma\, w^{\gamma(\frac{1}{2}-\beta)}.$$

Hence, for some constant C independent of m,

$$|\mathrm{A}_\delta^{(m)0}| \le \|\varphi\|_\infty \int_0^\delta \Big(\big(H_w^{*(m)}\big)^\gamma + \big(H_w^*\big)^\gamma\Big) L_w^{*(m)}\, dw \le C \int_0^\delta w^{\gamma(\frac{1}{2}-\beta)} L_w^{*(m)}\, dw.$$

Finally, by Cauchy–Schwarz and $\|L^{*(m)}\|_2 \le B$,

$$\int_0^\delta w^{\gamma(\frac{1}{2}-\beta)} L_w^{*(m)}\, dw \le \Big(\int_0^\delta w^{2\gamma(\frac{1}{2}-\beta)}\, dw\Big)^{1/2} \Big\|L_{(\cdot)}^{*(m)}\Big\|_2 \le B\Big(\int_0^\delta w^{2\gamma(\frac{1}{2}-\beta)}\, dw\Big)^{1/2}.$$

Since $\gamma(\frac{1}{2}-\beta) > 0$, the integral $\int_0^\delta w^{2\gamma(\frac{1}{2}-\beta)} dw$ vanishes as $\delta \downarrow 0$. Thus $\sup_m |\mathrm{A}_\delta^{(m)0}| \to 0$ as $\delta \downarrow 0$. Combining these bounds shows $\mathrm{A}^{(m)} \to 0$.

Therefore, for every $\varphi \in C([0,1])$,

$$\int_0^1 \varphi_w\, G_w^{(m)} dw \to \int_0^1 \varphi_w\, y_w^* dw.$$

Step 5.5 (Conclude $y_w^* = y$ a.e.) By Step 5.2, for every $\varphi \in C([0,1])$,

$$\int_0^1 \varphi_w \mathrm{g}_w^{(m)}\, dw \to y^* \int_0^1 \varphi_w\, dw.$$

By Steps 5.3 and 5.4, $\int \varphi_w g_w^{(m)} - \int \varphi_w G_w^{(m)} \to 0$ and $\int \varphi_w G_w^{(m)} \to \int \varphi_w y_w^*$, so

$$\int_0^1 \varphi_w\, y_w^*\, dw = y^* \int_0^1 \varphi_w\, dw \qquad \forall \varphi \in C([0,1]).$$

Since $y_{(\cdot)}^* \in \mathrm{L}^1(0,1]$, this implies $y_w^* = y^*$ a.e.

Step 6 (The equal-effective-labor allocation is optimal). Because $A_{(\cdot)}$ is always continuous and $L_{(\cdot)}^*$ produces equal effective labor across the interval, $L_{(\cdot)}^*$ must be continuous everywhere except at a removable set of discontinuity points. Let $\bar{L}_{(\cdot)}^*$ denote the continuous extension of $L_{(\cdot)}^*$. Because $\bar{L}_{(\cdot)}^* \in \mathcal{A}$ and is left-continuous, it must equal the unique equalizing allocation identified in Lemma 1.

Suppose by contradiction that a left-continuous allocation $L_{(\cdot)} \in \mathcal{A}$ yields $y > y^*$. Then by Lemma 3, it has a sequence of discrete approximations $L_{(\cdot)}^{(m)}$ with $y_{\bar{w}_m(w)}^{(m)} \to y_w > y^*$ a.e.

Let

$$\delta \equiv \left(1 - \left(\frac{y^*}{y}\right)^{\frac{1}{\gamma+1}}\right)/2.$$

Choose an m such that

i. $y^{*(m)} < 2y^*$ and

ii. $2^{-m}|K_m| < \left(\frac{\delta}{2}\right)^{\gamma+1}/(2y^*) \quad$ where $K_m \equiv \left\{k \,:\, \frac{y_k^{(m)}}{y^{*(m)}} \leq (1-\delta)^{-(\gamma+1)}\right\}$,

as we can by the fact that $y_{\bar{w}_m(w)}^{(m)} \to y_w \geq y > y^*$ a.e. and $y_{\bar{w}_m(w)}^{*(m)} \to y_w^* = y^*$ a.e., and the supposition that, by construction of δ, $y/y^* > (1-\delta)^{-(\gamma+1)}$. We must have $|K_m| \geq 1$, or else $y_k^{(m)} > y^{*(m)}$ for all k, contradicting the optimality of $y^{*(m)}$.

Let K_m^c denote K_m's complement. Define

$$\tilde{x}_k^{(m)} \equiv \begin{cases} (1-\delta)x_k^{(m)} + \delta/2, & k = 1; \\ (1-\delta)x_k^{(m)}, & k \in K_m^c \setminus \{1\}; \\ (1-\delta)x_k^{(m)} + (\delta/2)/|K_m|, & k \in K_m \setminus \{1\}. \end{cases} \tag{26}$$

This allocation is feasible because, by construction, the labor savings from cutting fraction δ of the labor in each bin total δ, of which half is allocated to bin 1 and half (or less, if $1 \in K_m$) to bins $k \in K_m \setminus \{1\}$. To show that $\tilde{y}_k^{(m)} > y^{*(m)}$ $\forall k$:

- $\tilde{y}_1^{(m)} > 2^m(\delta/2)^{\gamma+1}$, which $> 2y^*|K_m|$ by (ii), which $> y^{*(m)}$ by (i) and $|K_m| \geq 1$.
- Likewise, for $k \in K_m \setminus \{1\}$, we have a learning stock exceeding $\delta/2$, from the allocation to bin 1, and a labor allocation of at least $(\delta/2)/|K_m|$. So $\tilde{y}_k^{(m)} > 2^m(\delta/2)^{\gamma+1}/|K_m|$, which $> 2y^*$ by (ii), which $> y^{*(m)}$ by (i).
- For $k \in K_m^c \setminus \{1\}$, we have $\tilde{y}_k^{(m)} \geq y_k^{(m)}(1-\delta)^{\gamma+1}$: a multiplier of no less than $(1-\delta)^\gamma$ from the learning lost by the δ labor cutback, and of exactly $(1-\delta)$ from the labor cutback directly. So by definition of K_m, $y_k^{(m)}(1-\delta)^{\gamma+1} > y^{*(m)}$.

$\square$

The optimal allocation, disjoint case. Lemma 4 establishes that an optimal allocation of a worker over an interval (i) exists and (iii) takes the effective-labor-equalizing form found in Lemma 1. It follows that the optimal allocation in the disjoint case, found granting assumptions (i) and (iii), does indeed exist and take that form.

We will now prove assumption (ii) that the optimal allocation is disjoint.

The optimal allocation is disjoint. Suppose $\{L^n_{(\cdot)}\}$ is not disjoint. We will first construct, for any $\delta > 0$, a disjoint allocation $\{L^n_{(\cdot)}\}^{(\delta)}$ with $y^{(\delta)} \geq (1-\delta)y$. We will then find that there is another disjoint allocation $\{\hat{L}^n_{(\cdot)}\}^{(\delta)}$ and a $\xi > 1$ such that $\hat{y}_w^{(\delta)} \geq \xi(1-\delta)y$ or $\hat{y}_w^{(\delta)} \geq \xi y^{(\delta)}$. This will imply that for $\delta < 1 - 1/\xi$, $\hat{y}^{(\delta)} > y$. Finally, since, fixing y_w, Y_w is weakly higher on a disjoint allocation (by the convexity of $(\cdot)^{1+\alpha}$), this will imply that any optimal allocation is disjoint.

Given a set P, let $\mathcal{N}_P \equiv \{n : \mathbf{L}^n_P > 0\}$. Choose a $V = (\underline{v}, \overline{v}]$ such that $|\mathcal{N}_V| > 1$ and, for all $n \in \mathcal{N}_V$, $\mathbf{L}^n_{V_-} > 0$ and $\mathbf{L}^n_{V_+} > 0$, where $V_- \equiv (0, \underline{v}]$ and $V_+ \equiv (\overline{v}, 1]$. Let $\mathcal{P}^0$ be the partition of $(0,1]$ consisting of V_-, V, and V_+.

Let

$$\underline{w}^n \equiv \max\{w : \mathbf{L}^n_w = 0\}.$$

Given $\delta > 0$, further (infinitely) partition the interval as follows.

Let

$$z^n_{\underline{w};w} \equiv 1 - \frac{\left(\int_0^{\underline{w}} (w-v)^{-\beta} L^n_v \, dv\right)^\gamma}{A^n_w}, \tag{27}$$

$$Z^n(\underline{w};\overline{w}) \equiv \max_{w \in [\underline{w},\overline{w}]} z^n_{\underline{w};w}. \tag{28}$$

Observe that $Z^n(\cdot;\overline{w})$ is defined, by the continuity of $z^n_{(\cdot;\cdot)}$ and compactness of $[\underline{w},\overline{w}]$, and is strictly decreasing, with $Z^n(\underline{w}^n,\overline{w}) = 1$ and $Z^n(\overline{w},\overline{w}) = 0$ for all $\overline{w} > \underline{w}^n$. It is trivially right-continuous, by the continuity of $z^n_{(\cdot);w}$ and the fact that $z^n_{w;w} = 0$. To show that it is left-continuous, take an increasing sequence $v_k \to v_0$, and for each k, choose $w_k \in [v_k, \overline{w}]$ such that $z^n_{v_k;w_k} = Z^n(v_k;\overline{w})$. The sequence $\{w_k\}$ must have a convergent subsequence, so relabel the sequence so that $w_k \to w^*$. By continuity of $z^n_{(\cdot;\cdot)}$,

$$Z^n(v_k;\overline{w}) = z^n_{v_k;w_k} \to z^n_{v_0;w^*} \le Z^n(v_0;\overline{w}).$$

Since $Z^n(\cdot;\overline{w})$ is decreasing, the above must hold with equality.

Let

$$\delta_1 \equiv 1 - (1-\delta)^{1/3}.$$

For each n, let $v^n_0 = 1$, and then for $k = 0, 1, \ldots$ let

- $v^n_{k+1} = [w : Z^n(w; v^n_k) = \delta_1]$ (defined by the continuity of $Z^n(\cdot, v^n_k)$),
- $v^n_{k-} = \left(v^n_{k+1} + \text{argmax}_{w \in [v^n_{k+1}, v^n_k]} z_{v^n_{k+1};w}\right)/2$, and
- $v^n_{(k+1)^-} = [w : Z^n(w; v^n_{(k+1)^+}) = \delta_1]$.

(The resulting sequence will approach $\underline{w}^n$, but instead of proving this, we can simply note that if $v^n_k \to \tilde{w} > \underline{w}^n$, we may let $v^n_\omega = \tilde{w}$ and repeat the process indefinitely until $\underline{w}^n$ is approached.) Then let Q^n be the partition of $(0, 1]$ consisting of $(0, \underline{w}^n]$ and the subintervals, open below and closed above, separated by $\{v^n_k, v^n_{k-}\}$.

Let

$$\delta_2 \equiv (1-\delta)^{-\frac{1-2\beta}{3\beta^2}} - 1,$$

and let $\{q_\kappa\}$ denote the unique elements of $\{v^n_k, v^n_{k-}\}$ in descending order. Construct $\tilde{Q}^n$ by refining Q^n so that, for each $\kappa \ge 1$, $|\tilde{Q}| < \delta_2(q_\kappa - q_{\kappa+1})$ for all $\tilde{Q} \in \tilde{Q}^n \setminus [0, \underline{w}^n]$ with $\max(\tilde{Q}) \le q_{\kappa+1}$.

Construct $\mathcal{P}^n$ by refining $\tilde{\mathcal{Q}}^n$ so that

$$\frac{\inf_{w\in P} A_w^n}{\sup_{w\in P} A_w^n} > (1-\delta)^{1/3} \quad \forall P \in \mathcal{P}^n \setminus [0, \underline{w}^n]. \tag{29}$$

Finally, let $\mathcal{P} \equiv \bigwedge_{n=0}^{N} \mathcal{P}^n$. We will now reallocate labor within each $P \in \mathcal{P}$ so that the resulting allocation $\{L_{(\cdot)}^n\}^{(\delta)}$ is disjoint across workflows and at least $(1-\delta)$ as much effective labor is allocated to each workflow as under $\{L_{(\cdot)}^n\}$.

For each $P \in \mathcal{P}$, reallocate $\{\mathbf{L}_P^n\}$ into $|\mathcal{N}_P|$ sub-intervals as follows. Let $w_1 = \inf(P)$. Then for each $\nu = 1, ..., |\mathcal{N}_P|$, pick an $n \in \mathcal{N}_P$ arbitrarily from among those not yet chosen, and set $w_{\nu+1}$ such that $\mathbf{y}_{(w_\nu, w_{\nu+1}]} = \mathbf{y}_P^n$. Allocate $L^{n(\delta)}$ across $(w_\nu, w_{\nu+1}]$ in proportion to y_w subject to $\mathbf{L}_{(w_\nu, w_{\nu+1}]}^{n(\delta)} = \mathbf{L}_P^n$:

$$L_w^{n(\delta)} = y_w \times \frac{\mathbf{L}_P^n}{\mathbf{y}_{(w_\nu, w_{\nu+1}]}}. \tag{30}$$

Given n and $w > \underline{w}^n$, let κ_w^n index the boundary point of partition $\mathcal{Q}^n$ with $w \in (q_{\kappa_w^n+1}, q_{\kappa_w^n}]$. Given any reallocation $\{\hat{L}_{(\cdot)}^n\}$ that maintains $\hat{\mathbf{L}}_{\tilde{Q}}^n = \mathbf{L}_{\tilde{Q}}^n$ for all $\tilde{Q} \in \tilde{\mathcal{Q}}^n$, $\hat{A}_w^n$ can be lower-bounded by the value it would take in a reallocation in which (a) n's labor in $(q_{\kappa_w^n+2}, w]$ were discarded and (b) within every $\tilde{Q} \in \tilde{\mathcal{Q}}^n$ with $\max(\tilde{Q}) \le q_{\kappa_w^n+2}$, n's labor was shifted entirely to $\inf(\tilde{Q})$. Shift (a) multiplies A_w^n by at least $1-\delta_1$, by construction of $\mathcal{Q}^n$. For each $\tilde{Q} \in \tilde{\mathcal{Q}}^n$, shift (b) replaces $\int_{\tilde{Q}} (w-v)^{-\beta} L_v^n \, dv$ with

$$\int_{\tilde{Q}} \left(w - \inf(\tilde{Q})\right)^{-\beta} L_v^n \, dv \ge (1+\delta_2)^{-\beta} \int_{\tilde{Q}} (w-v)^{-\beta} L_v^n \, dv,$$

so multiplies n's productivity at w by at least $(1+\delta_2)^{-\frac{\beta^2}{1-2\beta}}$. In particular,

$$A_w^{n(\delta)} \ge (1-\delta_1)(1-\delta_2) A_w^n = (1-\delta)^{2/3} A_w^n. \tag{31}$$

Given w, let n be the unique worker with $L_w^{n(\delta)} > 0$, and let P be the unique element of $\mathcal{P}$ with $w \in P$. By (30) and (31),

$$y_w^{(\delta)} = A_w^{n(\delta)} L_w^{n(\delta)} \ge (1-\delta)^{2/3} A_w^n \, y_w \, \mathbf{L}_P^n / \mathbf{y}_P^n.$$

Since $\mathbf{L}_P^n/\mathbf{y}_P^n \geq 1/\sup_{v\in P} A_v^n$, by (29) we have

$$A_w^n \mathbf{L}_P^n/\mathbf{y}_P^n \geq (1-\delta)^{1/3}. \tag{32}$$

So $y_w^{(\delta)} \geq (1-\delta)y_w$.

Let $\sigma^{n(\delta)}$ denote the support of $L^{n(\delta)}$. Choose an $n \in \mathcal{N}_V$ with

$$\mu(\sigma^{n(\delta)} \cap V) < |V|/|\mathcal{N}_V|. \tag{33}$$

Let $\{\bar{L}_{(\cdot)}^n\}^{(\delta)}$ be a compression of $\{L_{(\cdot)}^n\}^{(\delta)}$ (i.e. an allocation with supports that partition the interval, maintaining the analog of (17)), and let upper bars indicate the corresponding supports, productivities, etc. $\bar{\bar{\sigma}}^{n(\delta)}$ will denote the support's maximum. WLOG assume that $\inf(\bar{\sigma}^n) = 0$.

Let $v^* \in \bar{\sigma}^{n(\delta)}$ denote the unique workflow with $\bar{\mathbf{L}}_{v^*}^{n(\delta)} = 1 - \mathbf{L}_{V_+}^n$, and let $\bar{V}_+ \equiv \big(v^*, \max(\bar{\sigma}_n^{(\delta)})\big]$. Observe that for each $w \in \bar{V}_+$, there is an $x(w) \in V_+$ with $\bar{L}_w^{n(\delta)} = L_{x(w)}^{n(\delta)}$ and $\bar{\mathbf{L}}_w^{n(\delta)} = \mathbf{L}_{x(w)}^{n(\delta)}$. Let

$$X \equiv \left(1 - \frac{1}{|\mathcal{N}_V|}\right)|V| \in (0,1).$$

Due to the compression across V in particular, we have for all $w \in \bar{V}_+$,

$$\begin{aligned}
\bar{H}_w^{n(\delta)} &\geq \int_0^{\inf(V)} \big(x(w) - v - X\big)^{-\beta} L_v^{n(\delta)} dv + \int_{\inf(V)}^{x(w)} \big(x(w)-v\big)^{-\beta} L_v^{n(\delta)} dv \\
&\geq (1-X)^{-\beta} \int_0^{\inf(V)} \big(x(w)-v\big)^{-\beta} L_v^{n(\delta)} dv + \int_{\inf(V)}^{x(w)} \big(x(w)-v\big)^{-\beta} L_v^{n(\delta)} dv \\
&\geq \big((1-X)^{-\beta} - 1\big) \int_0^{\inf(V)} \big(x(w)-v\big)^{-\beta} L_v^{n(\delta)} dv + H_{x(w)}^{n(\delta)} \\
&\geq \big((1-X)^{-\beta} - 1)\mathbf{L}_{V_-}^n + (1-\delta)^{\frac{2}{3\gamma}} H_{x(w)}^n \quad \text{by (31)} \\
&\geq \left[\frac{(1-X)^{-\beta}-1}{\max_w H_w^n} \mathbf{L}_{V_-}^n + (1-\delta)^{\frac{2}{3\gamma}}\right] H_{x(w)}^n \\
&\geq \big(\underline{X}^n\big)^{\frac{1}{\gamma}} H_{x(w)}^n, \qquad\qquad \underline{X}^n \equiv \left(1 + \frac{(1-X)^{-\beta}-1}{2\max_w H_w^n}\mathbf{L}_{V_-}^n\right)^{\gamma} > 1
\end{aligned}$$

$$\text{for} \quad \delta \le \overline{\delta}^n \equiv 1 - \left(2 - \left(\underline{X}^n\right)^{\frac{1}{\gamma}}\right)^{\frac{3\gamma}{2}}; \tag{34}$$

so $\bar{A}_w^{n(\delta)} \ge \underline{X}^n A_{x(w)}^n$; and so, because $\bar{L}_w^{n(\delta)} = L_{x(w)}^{n(\delta)}$,

$$\begin{aligned} \bar{y}_w^{(\delta)} &\ge \underline{X}^n L_{x(w)}^{n(\delta)} A_{x(w)}^n \\ &\ge \underline{X}^n (1-\delta)^{1/3} y \quad \text{by (30) and (32)}, \end{aligned}$$

where $\underline{X}^n$ is independent of δ. Thus $\bar{y}_w^{(\delta)} \ge (\underline{X}^n)^{1/2} y$ for

$$\delta \le \delta^n \equiv \min\left(\overline{\delta}^n, 1 - (\underline{X}^n)^{-\frac{3}{2}}\right). \tag{35}$$

Assume $\delta \le \delta^n$. Let $\chi^n \equiv 1 - (\underline{X}^n)^{-\frac{1}{4}\frac{1-2\beta}{1-\beta}} \in (0,1)$, and let

$$\hat{L}_w^{n(\delta)} = \begin{cases} \left(1 + \frac{\chi^n \mathbf{L}_{V_+}^n/2}{1-\mathbf{L}_{V_+}^n}\right) \bar{L}_w^{n(\delta)}, & w \le v^*; \\ (1-\chi^n)\bar{L}_w^{n(\delta)}, & w \in \bar{V}_+; \\ \xi_2^n y, & w \in \left(\bar{\bar{\sigma}}^{n(\delta)}, \bar{\bar{\sigma}}^{n(\delta)} + \Delta^n\right]; \\ 0, & w > \bar{\bar{\sigma}}^{n(\delta)} + \Delta^n, \end{cases}$$

where

$$\Delta^n \equiv \frac{\chi^n \mathbf{L}_{V_+}^n/2}{\xi_2^n y}, \quad \xi_2^n \equiv \left(1 - \chi^n \mathbf{L}_{V_+}^n/2\right)^{-\gamma} \Big/ \chi^n.$$

Multiplying labor within $\bar{V}_+$ by $1-\chi^n$ multiplies productivity throughout $\bar{V}_+$ by at least $(1-\chi)^{\frac{\beta}{1-2\beta}}$, yielding

$$\hat{y}_w^{(\delta)} \ge (1-\chi)^{\frac{1-\beta}{1-2\beta}} \bar{y}_w^{(\delta)} = \left(\underline{X}^n\right)^{-\frac{1}{4}} \bar{y}_w^{(\delta)} \ge \left(\underline{X}^n\right)^{\frac{1}{4}} y \;\; \forall w \in \bar{V}_+.$$

Of the $\chi \mathbf{L}_{V_+}^n$ units of labor saved by the reallocation from $\bar{V}_+$, half is allocated to $(0, v^*]$ in proportion to $\bar{L}^{n(\delta)}$. Letting ξ_1^n denote the coefficient on $\bar{L}_w^{n(\delta)}$ in the $w \le v^*$ interval, we have

$$\hat{y}_w^{(\delta)} \ge \xi_1^n y^{(\delta)} \; \forall w < v^*,$$

where $\xi_1^n > 1$ is independent of δ.

The other half is allocated to an interval above $\bar{\bar{\sigma}}^{n(\delta)}$. For all $w \geq \bar{\bar{\sigma}}^{n(\delta)}$ we must have $\hat{A}_w^{n(\delta)} \geq \left(1-\chi^n \mathbf{L}_{V_+}^n/2\right)^\gamma$, because the penalty to distant learning $(w-v)^{-\beta}$ cannot be less than 1 (at $w - v = 1$). So

$$\hat{A}_w^{n(\delta)} \hat{L}_w^{n(\delta)} > y/\chi^n \quad \forall w \in \left(\bar{\bar{\sigma}}^{n(\delta)}, \bar{\bar{\sigma}}^{n(\delta)} + \Delta^n\right],$$

where $1/\chi^n > 1$ is independent of δ.

For $m \neq n$, let $\{\hat{L}_{(\cdot)}^{m(\delta)}\}$ be an optimal uniform partition of $(\bar{\bar{\sigma}}^{n(\delta)} + \Delta^n, 1]$. Relative to an optimal uniform partition of $(\bar{\bar{\sigma}}^{n(\delta)}, 1]$—which would yield transformed output weakly greater than $\bar{y}_w^{(\delta)}$—this partition assigns each $m \neq n$ an interval at most $1 - \Delta$ times as large. So, by (14) and (15),

$$\hat{y}_w^{(\delta)} \geq (1 - \Delta^n)^{-(1+\gamma)(1-\beta)} \bar{y}^{(\delta)} \quad \forall w > \bar{\bar{\sigma}}^{n(\delta)} + \Delta^n,$$

where Δ is independent of δ.

Denote

$$\xi^n \equiv \min\left((\underline{X}^n)^{\frac{1}{4}}, \xi_1^n, 1/\chi^n, (1 - \Delta^n)^{-(1+\gamma)(1-\beta)}\right) > 1.$$

We have established that for $\delta \leq \delta^n$, $\hat{y}_w^{(\delta)} \geq \xi^n y$ or $\hat{y}_w^{(\delta)} \geq \xi^n y^{(\delta)} = (1 - \delta)\xi^n y$ for all w. So, if $\delta < \min(\delta^n, 1 - 1/\xi^n)$ for some n satisfying (33) given δ, $\hat{y}^{(\delta)} > y$. And given any $\delta > 0$, there is some $n \in \mathcal{N}_V$ satisfying (33). So for all $\delta < \min_n(\min(\delta^n, 1-1/\xi^n))$, $\hat{y}^{(\delta)} > y$.