Rank-Dependent Utility Theory and the Value of Evaluation

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Abstract

I propose a new axiom of normative decision theory, termed "Evaluation", and show that it is substantively weaker and perhaps more normatively compelling than the von Neumann-Morgenstern Independence axiom. I then observe that if a decision-maker obeys the axioms of rank-dependent utility (RDU) theory, and is further committed to the proposed principle that she should never actively refuse a costless opportunity to "fully evaluate" an act, she cannot be anywhere risk-avoidant. As RDU is generally used to justify risk-avoidant (rather than risk-inclined) deviations from expected-utility-maximizing behavior, this observation weakens the case for RDU as a normative decision theory.

1 Introduction

"The axioms should not be too numerous, their system is to be as simple and transparent as possible, and each axiom should have an immediate intuitive meaning by which its appropriateness may be judged directly."

– John von Neumann and Oskar Morgenstern
Theory of Games and Economic Behavior

We choose among the actions available to us in the face of uncertainty about what outcomes will result from them. In 1944, John von Neumann and Oskar Morgenstern famously proved that any agent whose choices in the face of uncertainty satisfy a short list of axioms can be represented as maximizing the expected value of some real-valued function, unique up to affine transformation, of final outcomes. This function is known as the agent's utility function, and the claim that rational agents should all be representable in this way is known as expected utility theory (EU).

For the past seventy-five years, decision theorists have debated this claim. The most widely accepted position is that the claim is correct. Another common position is that one of the von Neumann-Morgenstern axioms, termed Independence, should be replaced with some weaker axiom. Such replacements are designed to produce decision theories which permit some reasonable-seeming, but not expected-utility-maximizing, patterns of behavior as well.

As Chew and Epstein (1989) testify, and as a review of the subsequent literature confirms, there are two primary means of weakening Independence. The most commonly explored weakening is to an axiom known as Comonotonic Independence. This replacement generalizes expected utility theory to rank-dependent utility theory (RDU): a decision theory first sketched by Quiggin (1982) (under the name "anticipated utility theory"), refined by Yaari (1987), and popularized and normatively defended by Buchak (2013) (under the name "risk-weighted expected utility theory"). RDU permits various desired deviations from expected utility maximization, but it also comes with some undesirable features: in particular, it permits certain particularly counterintuitive reactions to information, as we will explore below.

The other commonly proposed weakening is to an axiom known as Betweenness. (See Dekel (1986) and Gul (1991) for examples of decision theories that satisfy the Betweenness axiom.) Chew and Epstein (1989) show that Betweenness and Comonotonic Independence together entail Independence. In other words, replacing Independence with Comonotonic Independence requires rejecting not merely Independence, but even Betweenness. Given EU's continued supremacy, many evidently find this too much to ask. Given the ongoing proposed defenses of RDU, on the other hand, many also evidently do not consider Betweenness to be universally valid. If we are to resolve the debate between these decision theories, therefore, we cannot appeal to Betweenness; we must appeal to the implications of weaker, more universally acceptable principles of rational behavior.

This paper proves that there is an axiom of normative decision theory, strictly weaker than Betweenness, which nevertheless (in conjunction with Comonotonic Independence) rules out precisely those desired deviations from EU-maximization that most strongly motivate the search for alternative decision theories. This proposed axiom, termed Evaluation, carries a straightforward intuitive meaning and obvious normative appeal.

2 Framework

We will use a slight modification of the Savage (1954) framework, in which an agent faces a set X of possible outcomes and is uncertain about the state of the world.

For simplicity, we will assume that the uncertainty about the state is representable by a standard continuous probability space $\mathscr{S} = (S, \mathcal{A}, \mu)$. We will denote the true state by s^*.

Definition 2.1. An <u>act</u> $F \in X^S$ is a function from S to X.

Definition 2.2. An <u>event</u> $\mathcal{E} \in \mathcal{A}$ is a measurable set of states.

Let Φ be the set of acts under consideration, where $\Phi = \{F \in X^S : |F(S)| < \infty, F^{-1}(x) \in \mathcal{A}\ \forall x \in X\}$. That is, for simplicity, we will only consider the agent's preferences over acts which realize a finite set of outcomes and which assign outcomes to events.

Definition 2.3. A <u>simple act</u> f is an act which assigns the same outcome to all states.

We will denote simple acts by lowercase letters and acts in general by uppercase letters, with $\phi \subset \Phi$ denoting the set of simple acts.

We will denote the probability distribution over X induced by some act F by $d(F)$, and that induced given some event $\mathcal{E}$ by $d(F|\mathcal{E})$. We will denote the p-mixture of an ordered pair of distributions (d_1, d_2) by $(p, d_1; 1-p, d_2)$. Note that $\exists H \in \Phi : d(H) = (p, d(F); 1-p, d(G))\ \forall F, G \in \Phi\ \forall p \in [0,1]$. That is, the set of distributions over X inducable by acts under consideration is closed under p-mixture.

We will denote the outcome of act F in state s by $x(F, s)$. For simplicity, we will sometimes denote the outcome of a simple act f by $x(f)$.

Definition 2.4. An agent's preferences over a set of acts Γ satisfy the Completeness axiom iff, for any two acts $F, G \in \Gamma$, $F \succsim G$ or $G \succsim F$.

Definition 2.5. An agent's preferences over some set of acts Γ satisfy the Transitivity axiom iff, for any three acts $F, G, H \in \Gamma$ where $F \succsim G$ and $G \succsim H$, $F \succsim H$.

Proposition 2.1. An agent's preferences over ϕ satisfy Completeness and Transitivity iff they can be represented by a utility function over outcomes $u : X \to \mathbb{R}$, unique up to strict monotonic transformation, such that $f \succsim g \iff u(x(f)) \geq u(x(g))\ \forall f, g \in \phi$.

A brief proof is given by Mas-Colell et al. (1995), p. 9.

Given a utility function u over outcomes, we will impose an ordering on the image $F(S)$ of each act F, where $i \leq j \to u(F(S)_i) \leq u(F(S)_j)\ \forall i, j \in I_{|F(S)|}$.

We will also assume "continuity in outcomes": that, for any two outcomes $x_1, x_2 \in X$ and any $p \in [0,1]$, $\exists x \in X : u(x) = pu(x_1) + (1-p)u(x_2)$. One might consider this analogous to the assumption of a continuous commodity space in consumer theory.

Finally, we will define the following:

Definition 2.6. The value of an act F, $v(F) = u(x(F, s^*))$, is the utility assigned to the outcome of the act in the true state.

Definition 2.7. An act F is admissible with respect to some set of acts $\Gamma \ni F$ iff $F \succsim G\ \forall G \in \Gamma$.

We will denote the set of acts admissible with respect to Γ by $b(\Gamma)$.

Definition 2.8. An agent's preferences over a set of acts Γ satisfy the Continuity axiom iff, for any three acts $F, G, H \in \Gamma$ where $F \succsim G \succsim H$, $\exists p \in [0,1] : G \sim G'\ \forall G' : d(G') = (p, F; 1-p, H)$.

Note that if an agent's preferences over Γ satisfy Continuity as stated above, then $d(F) = d(G) \to F \sim G\ \forall F, G \in \Gamma$.

We will assume that a rational agent's preferences over Φ satisfy Completeness, Transitivity, and Continuity.

3 Independence and EU

In addition to the three axioms of rational decision-making defined above, many find the following to have some intuitive appeal.

Definition 3.1. An agent's preferences over a set of acts Γ satisfy the Independence axiom iff, for any five acts $F, G, H, K, L \in \Gamma$ where $G \succsim F$, $d(K) = (p, d(F); 1-p, d(H))$, and $d(L) = (p, d(G); 1-p, d(H))$, $L \succsim K$.

3.1 Expected utility theory

Proposition 3.1 (The von Neumann-Morgenstern Utility Theorem)**.** An agent's preferences over Φ satisfy Completeness, Transitivity, Continuity, and Independence iff they can be represented by a utility function over outcomes $u : X \to \mathbb{R}$, unique up to positive affine transformation, such that $F \succsim G \iff \mathbb{E}[u(x(F,s))] \geq \mathbb{E}[u(x(G,s))]$.[1]

Informally, an agent's preferences over acts satisfy the von Neumann-Morgenstern axioms if and only if the agent *maximizes expected value.*

Definition 3.2. Expected utility theory is the claim that rational agents must maximize expected value.

An alternative axiomatization of expected utility theory can be found in Savage (1954). Savage does not assume that the state space comes endowed with a probability measure, as we have done. Instead, he proves that, as long as the agent's preferences over acts satisfy various axioms, there is a probability measure over the state space *and* a utility function over the outcome set with respect to which the agent maximizes expected value.

One of Savage's axioms is termed the "sure thing principle". As Friedman and Savage (1952) show, this principle, once formalized in the context of well-defined distributions over outcomes, is equivalent to Independence. We will therefore refer exclusively to the latter for the remainder of this paper.

3.2 The debate over Independence: background

These axiomatizations sparked an extensive debate over the validity of expected utility theory. This debate consisted primarily (as testifed by, e.g., Friedman and Savage (1952), p. 468) of a debate over the Independence axiom. It has remained so to this day.[2]

[1] Note that these expectations will always be well-defined because we are assuming that each act induces only finitely many outcomes. Also, note that von Neumann and Morgenstern's original (1944) proof only covered the case in which $|X|$ itself is finite. For an extension to the infinite case discussed here, see Kreps (1984), ch. 5.

[2] But see for example Tarsney (2018) for a proposed decision theory violating Completeness; Fishburn (1991) for a partial survey of decision theories violating Transitivity; or Arrhenius and Rabinowicz (2005) for a defense of violating Continuity.

The case against Independence is most clearly motivated by the observation that if expected utility theory is true, certain widely appealing act-preferences are irrational.[3] The most famous of these are described by Allais (1953).

Consider the following four acts, as described by their induced distributions over outcomes:

- If A is chosen, the agent receives \$0 with probability 0.01, \$1 million with probability 0.89, and \$5 million with probability 0.10.
- If B is chosen, the agent receives \$1 million for certain.
- If C is chosen, the agent receives \$0 with probability 0.90 and \$5 million with probability 0.10.
- If D is chosen, the agent receives \$0 with probability 0.89 and \$1 million with probability 0.11.

As Allais intuited, and as many researchers have experimentally confirmed[4], most people prefer B to A and C to D. These preferences, however, violate Independence. To see this, let x_1 denote the outcome of receiving \$0, x_2 denote receiving \$1 million, and x_3 denote receiving \$5 million, and let $d^* = (\frac{1}{11}, x_1; \frac{10}{11}, x_3)$. Now observe that

- $d(A) = (0.89, x_2; 0.11, d^*)$
- $d(B) = (0.89, x_2; 0.11, x_2)$
- $d(C) = (0.89, x_1; 0.11, d^*)$
- $d(D) = (0.89, x_1; 0.11, x_2)$

Independence requires that an agent's preferences between d^* and x_2 not depend on whether she faces a background probability of x_2 or x_1. In other words, Independence requires $A \succ B \iff C \succ D$.

In the face of appealing but Independence-violating preferences, decision theorists must weigh the normative appeal of the Independence axiom against that of the preferences in question. Weighing in favor of the former are arguments that violations of Independence inevitably produce (less ambiguously) irrational behavior. Weighing in favor of the latter are arguments that Independence can be replaced with weaker axioms which permit appealing preferences like those given above while prohibiting (less ambiguous cases of) irrationality.

A thorough history of the early debate over Independence is given by Fishburn and Wakker (1995). A more recent overview and extension of some of the relevant arguments can be found in Buchak (2013). A survey of the large literature on Independence lies outside the scope of this paper, but we will now outline the most relevant territory.

[3] It was also originally motivated by rudimentary analogies between probability mixtures and mixtures of commodities. See, for example, Manne and Charnes (1952) and Wold (1952) for early formulations of such arguments against Independence.

[4] See for instance Morrison (1967), Raiffa (1968), and Slovic and Tversky (1974).

3.3 Arguments for Independence

3.3.1 The "sure thing" argument

The earliest and simplest argument put forward in favor of Independence is what might be called the *sure thing argument*. The argument consists of an emphasis on the fact that Independence only requires an agent to prefer one act F to another G when, in the face of uncertainty about an arbitrary event, she would prefer F to G both if she knew that the event obtained and if she knew that it did not. That is, Independence only requires a preference of F to G when F is, in some sense, "surely" preferable to G.

More formally: consider acts $F, G, F_1, ..., F_n, G_1, ..., G_n$ and a measurable n-partition $\mathscr{P}$ on S, where $d(F|\mathscr{P}_i) = d(F_i)$, $d(G|\mathscr{P}_i) = d(G_i)$, and $G_i \succsim F_i \; \forall i \in I_n$. The sure thing argument maintains that rationality here *prima facie* requires $G \succsim F$.

The following illustration of this intuition is taken from Savage (1954, p. 21).

> A businessman contemplates buying a certain piece of property. He considers the outcome of the next presidential election relevant... [H]e asks whether he would buy if he knew that the Democratic candidate were going to win, and decides that he would do so. Similarly, he considers whether he would buy if he knew that the Republican candidate were going to win, and again finds that he would do so. Seeing that he would buy in either event... he should buy.

When presented this way, Savage then writes that "except possibly for the assumption of simple ordering, I know of no other extralogical principle governing decisions that finds such ready acceptance". Friedman and Savage (1952, p. 469) likewise guess that, after reflecting on the sure-thing reasoning above, a person will find that the Independence axiom "is not one he would deliberately violate".

At least a brief discussion of the sure thing argument for Independence can be found in almost any introduction to decision theory.

3.3.2 The "value of information" argument

Another argument for Independence follows from our intuitions about how a rational agent responds to information. In particular, it follows from the widespread impression that "the acceptance of costless perfect information is a fundamental property of rational behaviour in a single person game against nature setting; to reject costless information in such a setting seems self-evidently irrational" (Keasey 1984, p. 648).

More formally: consider $n+1$ k-sets of acts $\Gamma = \{F_1, ..., F_k\}, \Gamma_1 = \{F_{1,1}, ..., F_{1,k}\}, ..., \Gamma_n = \{F_{n,1}, ..., F_{n,k}\}$ and a measurable n-partition $\mathscr{P}$ on S, where $d(F_i|\mathscr{P}_j) = d(F_{i,j}) \; \forall i \in I_k, \forall j \in I_n$. Now consider an act G such that $d(G) = (\mu(\mathscr{P}_1), d(B_1); ...; \mu(\mathscr{P}_n), d(B_n))$, where $B_j \in b(\Gamma_j) \; \forall j \in I_n$. Observe that G can be interpreted as the act of acquiring information, as given by $\mathscr{P}$, before choosing from Γ. The value of information argument maintains that rationality here *prima facie* requires $G \succsim F \; \forall F \in \Gamma$.[5]

Blackwell (1953) proves that an expected utility maximizer will, as desired, always accept costless information in a standard single-person decision. Good (1967) independently proves the same (seeking in fact to *motivate* what he takes to be an obvious but unproven principle—that one should always accept costless information—from what he takes to be the more primitive principle that one should always maximize expected utility). Wakker (1988) proves the converse: that if an agent violates Independence, she will necessarily sometimes be information-avoidant. Wakker takes this to be strong evidence in Independence's favor.

Further testimony to the normative force of the "value of information" argument for Independence is given by Kadane et al. (2008), Al-Najjar and Weinstein (2009), and Bradley and Steele (2016), who explore the implications of weakening Independence to allow for "ambiguity aversion"; by Hilton (1990), who explores the implications of Machina's (1982) proposal to replace Independence with a weaker "preference-smoothness" condition; and by Briggs (2016), who explores the implications of rank-dependent utility. In each case, the author(s) find information-aversion to be an unacceptable consequence of the respective weakening of the Independence axiom.

3.3.3 Other arguments

The case for Independence is made on many grounds beyond the two outlined above. Foremost among these are arguments to the effect that violations of Independence produce undesirable behavior in the context of sequential choice problems. For example, it is sometimes claimed that violations of Independence require an agent to make sequences of decisions which, viewed collectively, produce distributions of outcomes which the agent disprefers to other distributions of outcomes which were available to the agent all along.

I will therefore close this section by noting that these "sequential choice" arguments for Independence can typically be defused by positing that a rational agent evaluates acts in a sequential choice situation not by the distributions of outcomes that will immediately result from each act in the sequence, but by the overall distributions of outcomes that each act induces when

[5] Outside this "game against nature" setting, it is easy to find situations in which one does best to reject costless information. Most straightforwardly, one should reject information when accepting it would prompt an adverse response from others. If one consults a genetic health test, for instance, one's health insurer may raise one's premium.

Perhaps these adverse responses should be understood as "costs" to acquiring the information in question, however. More subtly, then, consider "symptomatic" acts—those whose performance *predictably* allows the agent to rule out certain states. Symptomatic acts are impossible within the framework used here, which implicitly assumes that the chosen act is independent of the state; but they are undoubtedly possible in reality, and the two most common accounts of how to evaluate them both sometimes recommend information avoidance (as shown by Adams and Rosencrantz (1980) and Maher (1990), respectively). Some (e.g. Buchak (2013), p. 188) suggest that this weakens the "value of information" argument for Independence in general.

In any event—regardless of whether we can endorse a more general rule against rejecting costless information, we will restrict our discussion in this paper to the debate over the more limited and intuitive rule against rejecting costless information from within the Savage framework, as described above.

all is said and done. To use the terminology introduced by McClennen (1990) (and employed by Buchak (2013)), this is the position that rational agents are "sophisticated choosers".[6] I will assume this, and ignore the general question of sequential choice, for the remainder of this paper.

4 Comonotonic Independence and RDU

4.1 Arguments for weakening Independence to Comonotonic Independence

We have sketched two primary arguments in favor of the Independence axiom: "sure thing" and "value of information". Some claim that these arguments are not valid as universally as they might seem, and that we should reject each of them in favor of more restricted arguments, which might be called "comonotonic sure thing" and "value of comonotonic information". These arguments support a weaker axiom of rational behavior, termed Comonotonic Independence.

4.1.1 Weakening the "sure thing" argument

The "sure thing" argument holds that whenever the outcome-distribution induced by one act G is weakly preferred to that induced by another act F both if some event $\mathcal{E}$ obtains and if it does not, then we should find $G \succsim F$ in general.

As McClennen (1990, ch. 3.8) and Buchak (2013, ch. 5.4) have pointed out, however, the argument is typically motivated by illustrations in which we are led to believe that not merely the *sub-distribution* but in fact the *outcome* resulting from F is guaranteed to be preferred to that resulting from G. In McClennen's words, "when various writers have sought to 'motivate' or rationalize the independence principle, they have typically illustrated it with reference to the special case in which components are sure (riskless) outcomes" (p. 59). Consider Savage's businessman again, for example: we might have imagined that he knows (or thinks he knows) roughly what the property will be worth in the event of each candidate's victory.

In such a case, perhaps, an intuitively undeniable "sure thing" argument would hold. This would amount to the criterion that G be preferred to F if G *statewise dominates* F. When G does not statewise dominate F, however—when there are states in which the outcome of F is preferred—the strength of the "sure thing" intuition is less clear. One might prefer F in the face of uncertainty about $\mathcal{E}$; maintaining one's choice to perform F upon learning whether $\mathcal{E}$ is simply *not sure* to result in an outcome preferred to that which will obtain if one performs G. One might therefore question whether such cases deserve the "sure thing" label at all.

Weakening Independence all the way to the criterion of Statewise Dominance, however, would classify an undesirably wide range of preferences as rationally permissible. Instead, therefore, Buchak (with many others uncomfortable with a wholesale endorsement of the "sure thing" argument for Independence) endorses an intermediate argument, which might be called the

[6] See Buchak (2013, ch. 6-7) for a more thorough discussion of the debate along these lines.

"comonotonic sure thing argument". This argument holds that, if acts F and G "agree on which states lead to better outcomes", and if an agent would prefer G conditional both on some event obtaining and on its failing to obtain, *then* the agent is rationally required to prefer G. The argument justifies a principle introduced by Schmeidler (1989), termed Comonotonic Independence.[7]

Definition 4.1. A set of acts Δ is a comoncone if $u(x(F,s)) \succsim u(x(F,s')) \iff u(x(G,s)) \succsim u(x(G,s'))\ \forall F,G \in \Delta\ \forall s,s' \in S$.

That is, Δ is a comoncone if the acts in Δ order the states the same way, in terms of the utilities of the states' respective outcomes.

Definition 4.2. An agent's preferences over a set of acts Γ satisfy the Comonotonic Independence axiom iff, for any five acts $F,G,H,K,L \in \Delta \subseteq \Gamma$ where $G \succsim F$, $d(K) = (p, d(F); 1-p, d(H))$, $d(L) = (p, d(G); 1-p, d(H))$, and Δ is a comoncone, $L \succsim K$.

4.1.2 Weakening the "value of information" argument

The "value of information" argument holds that whenever an agent has the opportunity to gain costless information about the state of the world before deciding between F and G, he is rationally required to do so.

As Buchak (2010; 2013, pp. 195–200) has pointed out, however, this argument is typically motivated by illustrations in which we are led to believe that the information is guaranteed to lead us toward the act with the preferred outcome—or at least, guaranteed not to lead us away from it. In reality, information is often *misleading*, in the sense that it leaves us worse off, once our decision has been made, than we would have been without it. Consider, for example, the case given by Buchak (2010, pp. 85, 97):

> You are a shipowner. One day you are standing on the dock by your vessel, admiring the raging sea, when you notice that a small craft carrying nine people has capsized. Your ship can carry them all to safety, and if you do not rescue them, they will surely die. If you attempt to rescue them and your ship is not seaworthy, you will die along with them, but happily, you are almost certain that it is seaworthy. And even more happily, you have just enough time to perform a small test... testing for rot on a part of the ship that is especially prone to rot but has little to do with the structural integrity of the ship.

When we imagine scenarios like these, designed to highlight information's potential to mislead, the strength of the "value of information" intuition is less clear.

The criterion that one accept information whenever it runs no risk of being misleading would naturally be formalized as the criterion that, if $F \succsim G$, one accept information given by

[7] Hong and Wakker (1996) introduce the analogous Comonotonic Sure Thing Principle, constructed for the framework without objective probabilities.

partition $\mathscr{P}$ when the outcome-distribution induced by G conditional on some partition-element $\mathscr{P}_i$ is preferred to that induced by F conditional on $\mathscr{P}_i$ only if $u(x(G,s)) \succsim u(x(F,s))\ \forall s \in \mathscr{P}_i$.

As with the "sure thing" argument, however, the objection raised to this "value of information" argument seems to prove too much. Weakening Independence all the way to a criterion of "Accepting Non-misleading Information" would classify an undesirably wide range of preferences as rationally permissible. Again, therefore, Buchak endorses an intermediate argument: here, what might be called the "value of comonotonic information" argument. This holds that, even if information-avoidance is not irrational *in general*, it is irrational "when the possible information to be learned is partitioned into news that is of a similar nature" for all the acts under consideration (2013, p. 198). That is, information is always desirable when it is guaranteed not to be ambiguous across acts—when it is guaranteed either to be good news or to be bad news, with respect to the outcome that would result from any act. That way, if it is misleading about one act, it is misleading "in the same direction" about the others. It will, in some sense, not be misleading "about *the choice between*" acts.

It will be evident on reflection that this argument precisely justifies Comonotonic Independence.

4.2 Rank-dependent utility theory

Wakker, Erev, and Weber (1994) show that weakening Independence to Comonotonic Independence, and keeping the other axioms in place, is equivalent to Yaari's (1987) formalization of "rank-dependent utility".

Proposition 4.1 (A representation theorem for RDU)**.** An agent's preferences over Φ satisfy Completeness, Transitivity, Continuity, and Comonotonic Independence iff they can be represented by a utility function over outcomes $u : X \to \mathbb{R}$, unique up to positive affine transformation, and a monotonically increasing risk function over probabilities $r : [0,1] \to [0,1]$ with $r(0) = 0$ and $r(1) = 1$, such that $F \succsim G \iff u(F(S)_1) + \sum_{i=2}^{|F(S)|} r(\mu(F^{-1}(F(S)_i)))(u(F(S)_i) - u(F(S)_{i-1})) \geq u(G(S)_1) \sum_{i=2}^{|G(S)|} r(\mu(G^{-1}(G(S)_i)))(u(G(S)_i) - u(G(S)_{i-1}))$.[8]

Definition 4.3. Rank-dependent utility theory is the claim that rational agents' preferences over acts must be representable as above.

Given utility function u and risk function r, we will denote the rank-dependent utility of an act F by $\text{RDU}_{u,r}(F) = u(F(S)_1) + \sum_{i=2}^{|F(S)|} r(\mu(F^{-1}(F(S)_i)))(u(F(S)_i) - u(F(S)_{i-1}))$.

Some appealing features of rank-dependent utility, such as the fact that it forbids the preference of stochastically dominated outcome-distributions, are identified by Quiggin (1982, 1993) and Yaari (1987) in originally introducing the theory. Wakker (1990) and Nakamura (1995) later find that RDU is roughly equivalent to "Choquet expected utility" (CEU), a decision theory developed to represent "ambiguity aversion" in the face of imprecise credences. Further discussion

[8]Proof of this representation in the infinite-outcome case is given by Abdellaoui (2002).

of the appealing features of rank-dependent utility lies outside the scope of this paper; Buchak (2013) offers a thorough exploration.

4.3 Risk-avoidance

Definition 4.4. An RDU-maximizing agent is anywhere-risk-avoidant if, given his risk function r, $r(p) < p$ for some $p \in (0,1)$.

Risk-avoidance can rationalize the Allais preferences. In particular, observe that an RDU-maximizer exhibits the Allais preferences iff $\frac{r(0.1)}{1+r(0.1)-r(0.99)} < \frac{u(x_2)-u(x_1)}{u(x_3)-u(x_1)} < \frac{r(0.1)}{r(0.11)}$ (where, as above, x_1 denotes the outcome of receiving \$0, x_2 denotes the outcome of receiving \$1 million, and x_3 denotes the outcome of receiving \$5 million). These inequalities are satisfied, for example, when $r(p) = p^2$, $u(x_1) = 0$, $u(x_2) = 1$, and $u(x_3) = 2$.[9]

Segal (1987) identifies convexity in the risk function as a necessary ingredient for Allais-like preferences more generally.[10] He finds that convexity is also necessary to explain preferences that exhibit the "common ratio effect"—another commonly observed deviation from expected utility maximization.

Finally, the "cumulative prospect theory" (CPT) of Tversky and Kahneman (1992) is by far the most widely accepted formalization of real-world human behavior in response to risk. CPT, once fitted to the experimental data, is equivalent to RDU with an "inverse-S-shaped" risk function and a privileged status-quo point, where $r(p) = p$ at the point where an act under consideration begins to offer some chance of "gains" rather than merely greater or lesser losses (Buchak (2013, pp. 59, 66)). That is, people are observed to exhibit risk-inclination with respect to possible losses, but risk-avoidance with respect to possible gains.

In sum, the ability to accommodate at least some form of anywhere-risk-avoidance is widely understood to be a necessary feature of any well-motivated generalization of expected utility theory.

5 Betweenness

As discussed in the Introduction, the other commonly proposed weakening of Independence is an axiom termed Betweenness.

Definition 5.1. An agent's preferences over a set of acts Γ satisfy the Betweenness axiom iff, for any three acts $F, G, H \in \Gamma$ where $H \succsim F$ and $d(G) = (p, d(H); 1-p, d(F))$ for some $p \in [0,1]$, $G \succsim F$.

[9] This example is given by Buchak (2013, p. 71).

[10] He refers to "concavity in the weighting function" rather than "convexity in the risk function", because he uses the formalization of RDU given by Yarri (1987) (and others), whereas we are using that given by Buchak (2013) (and others). The two conditions are equivalent, however: this follows immediately from Buchak (2013, p. 57).

The appeal of Betweenness is its status as a simple, intuitively compelling axiom which is weak enough to permit risk-avoidance (as reflected e.g. by the Allais preferences) but strong enough, in conjunction with the other von Neumann-Morgenstern axioms, to forbid a wide range of seemingly irrational behavior. Other weakenings generally either fail to permit the Allais preferences (see e.g. the Homotheticity axiom of Burghart, Epper, and Fehr (2014)) or go so far as to permit more objectionable preferences, such as preferences for stochastically dominated utility-distributions (see e.g. Karmarkar's (1978, 1979) theory of "subjectively weighted utility").

6 Evaluation

Definition 6.1. An agent's preferences over a set of acts Γ satisfy the Evaluation axiom iff, for any pair of acts and single simple act $F, G, h \in \Gamma$ where $h \succ F$ and $d(G) = (p, x(h); 1-p, d(F))$ for some $p \in [0,1]$, $G \succsim F$.

Note that, as stated here, Evaluation is simply Betweenness with the requirement that H be simple. Betweenness-satisfying preferences thus satisfy Evaluation. As we will see below, however, the converse does not hold; Evaluation is strictly weaker than Betweenness.

Proposition 6.1. In the context of any decision theory that forbids the preference of stochastically dominated outcome-distributions, Evaluation is equivalent to the following more general condition: For any pair of acts and set of simple acts $F, G, h_1, ..., h_n$ where $h_i \succ F \ \forall i \in I_n$ and $d(G) = (p_1, x(h_1); ...; p_n, x(h_n); 1 - \sum_{i=1}^n p_i, d(F))$ for some $p_1, ..., p_n \in [0,1]$, $G \succsim F$.

Proof: The backward implication is trivial. The forward implication follows from the fact that Evaluation implies that $\underline{G} \succsim F$ when $d(\underline{G}) = (\sum_{i=1}^n p_i, \underline{h}; 1 - \sum_{i=1}^n p_i, d(F))$, where $\underline{h}$ is a least preferred simple act among $\{h_1, ..., h_n\}$, and the fact that G stochastically dominates $\underline{G}$. ■

We can interpret Evaluation as the condition that, if one has the opportunity to costlessly learn the outcome of—or, "evaluate"—some act H (without learning anything about the outcome of F) before deciding between H and F, one should not prefer to reject this opportunity.

6.1 Arguments for Evaluation

On both lines of reasoning explored above, Evaluation stands out as a natural weakening of Independence which more naturally addresses the objections raised by defenders of Comonotonic Independence than does Comonotonic Independence itself.

6.1.1 The "sure thing" argument

The "sure thing" motivation for Evaluation is simple. If there is an event $\mathcal{E}$ such that the outcome of G is preferred to the outcome distribution of F if $\mathcal{E}$ obtains, and such that the outcome-distribution of G is identical to that of F if $\mathcal{E}$ does not obtain, it seems natural to say that G is surely preferable to F. More precisely, it seems like a natural step between mere statewise dominance and the "sub-act dominance" condition codified as Independence. G improves on F, the argument goes, by assigning each state in $\mathcal{E}$ to a known outcome preferred to act F as a whole.

Comonotonic Independence, by contrast, requires rational preferences over acts to obey full "sub-act dominance"—but only when all the acts in question are comonotonic. As several authors have noted (e.g. Luce 1996a, 1996b; Safra and Segal 1998), it is difficult to find a natural explanation of why this particular restriction would be desirable. Some (e.g. Diecidue and Wakker 2001) attempt one; but I expect most readers, including those most sympathetic to Comonotonic Independence, will not find these as natural as the justification for Evaluation.

6.1.2 The "value of information" argument

Recall the argument that it may be rational to avoid information out of fear that the information will be misleading—that it may lead us to consider a lottery more or less valuable than it in fact is. Even if we accept this argument, we may still be inclined to accept the Evaluation axiom, since it only requires a rational agent to accept information that will pin down the value of one act with certainty and shed no light on any other act. Turning down information may sometimes strike us as appealing, but doing so even when the information comes with *no possibility of misleading* us about any of the acts available to us appears especially troublesome.

Comonotonic Independence, by contrast, requires us to accept information whenever it is "of a similar nature" for the acts under consideration (i.e. "good news" or "bad news"). There is simply no straightforward sense in which this restricts the potential of the news to be misleading. As the reader can easily verify, for any act F that does not guarantee a best outcome, there is a comonotonic act H, an act G, an information partition $\mathscr{P}$, and a state s such that $d(G) = d(H|\mathscr{P}(s))$, $G \succ F$, and $H(s) = G(s) = x_1$. That is, except in the special case that F guarantees a best outcome, accepting comonotonic information can always mislead an agent into choosing an act with a *worst* outcome. By contrast, accepting the opportunity to evaluate H can never result in an outcome dispreferred to F, and therefore that it can never result in an outcome worse than the worst outcome possible under F.

Finally, one might point out that, even if Evaluation is more promising than Comonotonic Independence as a rational requirement in response to information, Evaluation still (like Comonotonic Independence) exposes an agent to the risk of switching from F to an act with a worse outcome (in the event that $v(F) > v(H)$). One might argue that Evaluation too, therefore, fails in some sense to protect the agent from misleading information. In response, rather than parse the word "misleading", recall the observation that the "sure thing" argument for Independence is typically illustrated by cases in which not merely the outcome-distribution, but the outcome, of F is surely preferred to G regardless of whether an event $\mathcal{E}$ obtains. The implication, in this instance, is that the case for accepting Independence over arbitrary sub-acts is weaker than we had been led to believe. Likewise, then, observe that the argument for rejecting misleading information is typically illustrated (Buchak 2010; 2013, p. 193) by cases in which one has the opportunity to gain some information about the outcome-distribution for an act, but not to fully identify its outcome. Partially examining the rot on one's ship runs the risk that one will let the strangers needlessly drown; *fully* determining the ship's seaworthiness, and thus the outcome of attempting a rescue, seems like a much stranger opportunity to dismiss. This, I submit, suggests that the case for rejecting *perfect* information about acts is *weaker* than we may have been led to believe. In other words, Evaluation does not ask us to do what troubles us when we seek to avoid misleading information.

A more thorough debate along these lines may be possible. For now we will simply trust that the force of the intuition for Evaluation is clear, and hope that it is felt by at least a few who are sympathetic to Comonotonic Independence but not generally persuaded of the obligation to maximize expected value.

7 Results

We can now state the following:

Proposition 7.1. An RDU-maximizer violates the Evaluation axiom if her risk function is anywhere risk-avoidant.

Proof: Consider by contradiction a somewhere-risk-avoidant RDU-maximizer with utility function u and risk function r. Choose $p \in (0, 1)$ such that $r(p) < p$ and the derivative $y = r'(p)$ is defined.

Since r is monotonic, it is differentiable almost everywhere (by Lebesgue's Theorem for the Differentiability of Monotone Functions). Since we know that $r(p^*) < p^*$ for some p^*, by definition of anywhere-risk-avoidance, it follows that we can find p as desired.

Define an act F such that

$$d(F) = (1 - p, x_1; p, x_2), \tag{1}$$

where $u(x_1) = 0$ and $u(x_2) = 1$.

Because $r(p) < p$, we can choose $\varepsilon > 0$ such that

$$\varepsilon < y\left(1 - \frac{r(p)}{p}\right). \tag{2}$$

Given ε, choose $\delta_1 > 0$ such that

$$\left|\frac{r(p) - r(p - \delta')}{\delta'} - y\right| < \varepsilon \ \forall \delta' \leq \delta_1. \tag{3}$$

Likewise, choose $\delta_2 > 0$ such that

$$\left|\frac{r(p + \delta') - r(p)}{\delta'} - y\right| < \varepsilon \ \forall \delta' \leq \delta_2. \tag{4}$$

Choices of δ_1 and δ_2 so as to satisfy (4) and (5), respectively, are possible by definition of the (two-sided) differentiability of r at p, with $r'(p) = y$.

Choose δ such that

$$\delta = \min(\delta_1, \delta_2). \tag{5}$$

Given δ, choose $q > 0$ such that

$$q < \min\left(\frac{\delta}{p}, \frac{\delta}{1-p}\right). \tag{6}$$

From (5) and (6), we have

$$qp < \delta_1,\ \ q(1-p) < \delta_2. \tag{7}$$

On substituting qp as δ' into (3), substituting $q(1-p)$ as δ' into (4), and rearranging, we have

$$r((1-q)p) < r(p) + (y-\varepsilon)(-qp), \tag{8}$$

$$r(q+(1-q)p) < r(p) + (y+\varepsilon)(q-qp). \tag{9}$$

By (6), $qp < \delta$ and $q - qp < \delta$. Therefore

$$r((1-q)p) < r(p) + y(-qp) + \varepsilon\delta, \tag{10}$$

$$r(q+(1-q)p) < r(p) + y(q-qp) + \varepsilon\delta. \tag{11}$$

Rearranging (2), we have

$$[r(p) + y(q-qp) + \varepsilon\delta]r(p) + [r(p) + y(-qp) + \varepsilon\delta](1-r(p)) < r(p). \tag{12}$$

Substituting the respective terms from (10) and (11), we have

$$r(q+(1-q)p)r(p) + r((1-q)p)(1-r(p)) < r(p). \tag{13}$$

Choose $\gamma \in (0, 1-r(p))$ such that

$$\gamma < \frac{r(p) - (r(p)r(q+(1-q)p) + (1-r(p))r((1-q)p))}{r(q+(1-q)p) - r((1-q)p)}. \tag{14}$$

Note that the numerator is strictly positive, by (13), and that the denominator is also strictly positive, by the strict monotonicity of r. Now, by construction of γ, we have

$$r(q+(1-q)p)(r(p)+\gamma) + r((1-q)p)(1-(r(p)+\gamma)) < r(p). \tag{15}$$

Let $m = r(p) + \gamma$, so that

$$r(q+(1-q)p)m + r((1-q)p)(1-m) < r(p). \tag{16}$$

Let x be an outcome such that $u(x) = m$ (as must exist since $m \in [u(x_1), u(x_2)]$), and let h be a simple act such that $x(h) = x$. Also, define an act G such that $d(G) = (q, x; 1-q, d(F))$. Observe that

$$\mathrm{RDU}_{u,r}(h) = m > r(p), \tag{17}$$

$$\mathrm{RDU}_{u,r}(F) = r(p), \tag{18}$$

$$\mathrm{RDU}_{u,r}(G) = r(q + (1-q)p)m + r((1-q)p)(1-m). \tag{19}$$

We have an act G, a simple act h, and a probability q such that $h \succ F$ (by (17), (18)), $d(G) = (q, x(h); 1-q, F)$, and $G \prec F$ (by (16), (18), (19)). Our agent thus violates the Evaluation axiom. ■

Note that we have let F be an arbitrary two-outcome act, offering the preferred outcome with any probability p so long as $r(p) < p$ and $r'(p)$ is defined; and, given this lottery, we have discovered that there is an evaluation opportunity which our RDU-maximizer would turn down. Thus, while we have shown that an anywhere-risk-avoidant RDU agent *sometimes* violates the Evaluation axiom, we have also shown something stronger: that if an RDU agent is anywhere-risk-avoidant—including, of course, if he has an everywhere-strictly-convex risk function—then, whenever he faces a pair of possible outcomes, there is a possible alternative that he would prefer to the lottery before him, but which he would avoid the chance to discover. That is, Evaluation-violating behavior is not restricted to curious, artificially constructed edge cases. There is a sense in which a globally risk-avoidant agent rejects evaluation opportunities *pervasively.*

Now, we will show that Evaluation permits a wide range of rank-dependent deviations from expected utility maximization. In doing so, we will demonstrate that Evaluation is in fact substantively weaker than Betweenness, and not merely a new rhetorical justification for a principle of rational behavior which defenders of RDU theory have already knowingly rejected.

Proposition 7.2. An RDU-maximizer with utility function u and risk function r obeys the Evaluation axiom if r is twice differentiable and everywhere concave.

Proof: Let r be twice differentiable and everywhere concave, and let F be an arbitrary act where

$$d(F) = (p_1, x(F)_1; ...; p_i, x(F)_i; ...; p_n, x(F)_n). \tag{20}$$

Let $x(F)_i$ denote an outcome with $u(x(F)_i) = \mathrm{RDU}_{u,r}(F)$. That is, if F offers any probability of this outcome, this probability is p_i. If not, $p_i = 0$. Now, given arbitrary $q \in [0, 1]$, let

$$d_F(q) = (q, x(F)_i; 1-q, d(F)), \tag{21}$$

or, expanded,

$$d_F(q) = ((1-q)p_1, x(F)_1; ...; (1-q)p_i + q, x(F)_i; ...; (1-q)p_n, x(F)_n). \tag{22}$$

From this we have

$$\begin{aligned}\mathrm{RDU}_{u,r}(d_F(q)) = {} & r((1-q)(p_1 + ... + p_n) + q)u(x(F)_1) \\ & + r((1-q)(p_2 + ... + p_n) + q)(u(x(F)_2) - u(x(F)_1)) + ... \\ & + r((1-q)(p_i + ... + p_n) + q)(u(x(F)_i) - u(x(F)_{i-1})) + ... \\ & + r((1-q)p_n)(u(x(F)_n) - u(x(F)_{n-1})).\end{aligned} \tag{23}$$

Taking the second derivative with respect to q,

$$
\begin{aligned}
(\mathrm{RDU}_{u,r} \circ d_F)''(q) = (1 - (p_1 + ... + p_n))^2\, r''((1-q)(p_1 + ... + p_n) + q)u(x(F)_1) \\
+ (1 - (p_2 + ... + p_n))^2\, r''((1-q)(p_2 + ... + p_n) + q)(u(x(F)_2) - u(x(F)_1)) + ... \\
+ (1 - (p_i + ... + p_n))^2\, r''((1-q)(p_i + ... + p_n) + q)(u(x(F)_i) - u(x(F)_{i-1})) + ... \\
+ (-p_n)^2\, r''((1-q)x(F)_n)(u(x(F)_n) - u(x(F)_{n-1})). \quad (24)
\end{aligned}
$$

Since r'' is always nonpositive (by r's concavity), the above expression is always nonpositive. That is, $\mathrm{RDU}_{u,r} \circ d_F$ is concave in q. And since $(\mathrm{RDU}_{u,r} \circ d_F)(0) = (\mathrm{RDU}_{u,r} \circ d_F)(1) = \mathrm{RDU}_{u,r}(F)$, we know that, for any act G such that $d(G) = (q, x(F)_i; 1-q; d(F))$ for some $q \in (0,1)$, $G \succsim F$.

Finally, since RDU forbids the preference of stochastically dominated outcome-distributions, it follows that, for any pair of acts and simple act F, G, h such that $h \succ F$ and $d(G) = (q, x(h); 1-q; d(F))$ for some $q \in [0,1]$, $G \succsim F$. RDU agents with concave, twice-differentiable risk functions thus obey the Evaluation axiom. ■

8 Conclusion

We have defined a new candidate axiom for normative decision theory, termed Evaluation. We have demonstrated that it is substantively weaker than the Independence axiom to which critics of expected utility theory commonly object, and strictly weaker even than the Betweenness axiom to which Independence is often weakened. We have argued that this weaker axiom is particularly normatively compelling because it avoids one of the main objections leveled at Independence: it does not ask an agent to risk exposing herself to "misleading" information about any of the acts available to her, but only requires her to act as she would upon being fully informed with respect to some act's value. Finally, we have proven that accepting this axiom, in the context of rank-dependent utility theory, forbids risk-avoidant behavior.

If a proponent of RDU theory comes to endorse the Evaluation axiom, she can accept the unusual conclusion that some risk-inclined behavior is rational but risk-avoidance is not. As we have seen, however, a desire to endorse risk-avoidance is what generally motivates weakening the Independence axiom, and developing alternatives to expected utility theory, in the first place. The weak constraint imposed by Evaluation, then, may on its own be enough to rule out all plausible non-linear risk functions, and return the RDU proponent to the fold of expected utility theory.

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